PRIME SPECIALIZATION IN HIGHER GENUS II

BRIAN CONRAD, KEITH CONRAD, AND ROBERT GROSS

Abstract. We continue the development of the theory of higher-genus Möbius periodicity that was studied in Part I for odd characteristic, now treating asymptotic questions and the case of characteristic 2. The extra difficulties in characteristic 2 are overcome via rigid geometry in characteristic 0. The results on Möbius periodicity in any positive characteristic are used to incorporate a correction factor into the false naive conjecture of Bateman–Horn type concerning how often a polynomial with a higher-genus coefficient ring takes prime values; numerical evidence is provided to support the suitability of this correction factor. We also prove some asymptotic and non-triviality properties of the correction factor.

1. Introduction

Let κ be a finite field with characteristic $p > 0$, and let $C = \operatorname{Spec} A$ be a smooth and geometrically connected affine curve over κ with exactly one geometric point ξ at infinity (so ξ is κ-rational). Let $K = \kappa(C)$ be the fraction field of A. For $a \in A - \{0\}$, let

$$\deg(a) := -\operatorname{ord}_\xi(a) \geq 0. \tag{1.1}$$

For $f \in A[T]$ that is irreducible in $K[T]$, it is natural to ask how often the ideal $(f(a))$ in A is prime as $\deg(a) \to \infty$. This is only interesting when $f(T) \in A[T]$ has *no local obstructions*: for every maximal ideal $\mathfrak{m}$ of A the function $f : A/\mathfrak{m} \to A/\mathfrak{m}$ is not identically zero. There is a standard conjecture that applies to this setting, as well as to the more general case when A is replaced by any ring of S-integers in a global field (with finite S), and in the case of number fields the numerical evidence looks favorable (even when the class group is non-trivial). We shall be interested in the case $f \in A[T^p]$ because in this case (in striking contrast with what is expected for K-separable f) the statistical properties of $\mu(f(a))$ are often nonrandom. This seems to influence the likelihood that $(f(a))$ is prime and gives counterexamples to the standard conjecture. Such nonrandomness in the case of genus 0 was used in [4] to construct some 1-parameter families of elliptic curves with surprising root number variation, and the main purpose of this paper is to prove and apply precise statistical properties of $\mu(f(a))$ as a varies in a higher-genus coordinate ring.

The study of $\mu(f(a))$ in odd characteristic was taken up in [5] for any $f = \sum_i \alpha_i T^i \in A[T^p]$ with $\deg_T f > 0$ such that f is squarefree in $K[T]$ and f is primitive with respect to A in the sense that $Z_f = \operatorname{Spec}(A[T]/(f)) \subseteq C \times \mathbf{A}^1_\kappa$ is quasi-finite over C. Assume in addition

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that f has no local obstructions. Let J be a nonzero ideal of A and consider the function

$$\overline{\mu}_{f,\kappa,J}(n) = \frac{\sum\limits_{\deg a=n,(f(a),J)=1} \mu(f(a))}{\sum\limits_{\deg a=n,(f(a),J)=1} |\mu(f(a))|}. \tag{1.2}$$

By [13, Thm. 8.1] (applied to a multiple of f by an element of A with order exactly 1 at each prime factor of J), if n is large enough then the denominator in this average is not zero. The method used in the proof of Lemma 2.1 below shows that such ineffective largeness for n can be improved *a posteriori* to only depend on the genus g of K/κ, the dimension $\dim_\kappa(A/J)$, and the *total degree*

$$\deg_{u,T} f := \max_i(-\operatorname{ord}_\xi(\alpha_i) + i).$$

In Theorem 3.1 we will use the results in [5] to prove the surprising fact that for $p \neq 2$ the complicated-looking average function $\overline{\mu}_{f,\kappa,J}$ is a function of $n \bmod 4$ when n is sufficiently large (largeness only depending on the genus, the total degree $\deg_{u,T} f$, and $\dim_\kappa(A/J)$), and that if $-1 \in \kappa^\times$ is a square or $\deg_T f$ is even then it only depends on $n \bmod 2$ for such large n. Beware that the periodic function of large n defined by (1.2) may change if we work with the scalar extension of the same data A, f, and J over a finite extension of κ; see [3, Ex. 6.6, 6.7].

To explain why such mod-4 periodicity is interesting, let $I_f \subseteq A$ be the nonzero radical ideal such that $\operatorname{Spec}(A/I_f) \subseteq C$ is the image in C of the finite branch scheme B for the generically étale projection $Z_f \to \mathbf{A}^1_\kappa$. (This projection is generically étale by [5, Lemma 2.2].) The dimension $\dim_\kappa(A/I_f)$ may be bounded above in terms of $\deg_{u,T} f$ and the genus (see the end of the proof of Lemma 2.1). Our interest in (1.2) is due to the fact that for $p \neq 2$ the *periodic* function $1 - \overline{\mu}_{f,\kappa,I_f}(n)$ for large n (and a variant in characteristic 2) appears to be the right correction factor to fix the false standard conjecture of Bateman–Horn type concerning primality statistics for the ideal $(f(a)) \subseteq A$ with $f \in A[T^p]$ and $a \in A$ satisfying $\deg a = n \to \infty$. This conjecture is formulated in §2, and in the appendix we test it numerically for examples drawn from curves with genus 1 and 2; the numerics work out well. We do not expect "Möbius periodicity" to occur for polynomials not in T^p. In Theorem 3.8, for odd p we prove (in a suitable sense) that $\overline{\mu}_{f,\kappa,I_f}(n)$ for large n is often not identically zero.

Since $1 - \overline{\mu}_{f,\kappa,I_f}(n)$ is being proposed as a correction factor (when $p \neq 2$) in a conjecture whose formulation over number fields does not appear to require a correction factor, it is natural to ask if, as we vary f or vary κ (with fixed $f \in A[T^p]$ that is squarefree in $K[T]$ and primitive over A), the function $\overline{\mu}_{f,\kappa,I_f}$ on $\mathbf{Z}/4\mathbf{Z}$ is close to 0. Our main result in this asymptotic direction is:

Theorem 1.1. *Assume $p \neq 2$ and let $f \in A[T^p]$ and $B \subseteq Z_f$ be as above. Choose $c \in \{0,1,2,3\}$ and let $\overline{\mu}_{f,\kappa,c}$ be the common value of $\overline{\mu}_{f,\kappa,I_f}(n)$ for sufficiently large n with $n \equiv c \bmod 4$ (where the largeness depends only on the genus and $\deg_{u,T} f$).*

As $[\kappa' : \kappa] \to \infty$, $\overline{\mu}_{f,\kappa',c}$ tends to 0, 1, or -1. Moreover, if the branch scheme B has odd length at some point then $\overline{\mu}_{f,\kappa',c} \to 0$ as $[\kappa' : \kappa] \to \infty$ for all $c \in \{0,1,2,3\}$.

In §3 (resp. §6) we give the proof of Theorem 1.1 (resp. the proof of the analogue of Theorem 1.1 for $p = 2$). By Theorem 3.6 below, for $p \neq 2$ and $f \in A[T^p]$ generic in suitable algebraic families of polynomials there exists some $x \in B$ such that $\ell(\mathscr{O}_{B,x})$ is odd (and even a power of p). Hence, for such "generic" f we have $1-\overline{\mu}_{f,\kappa',c} \to 1$ as $[\kappa':\kappa] \to \infty$ for all $c \in \{0,1,2,3\}$ when $p \neq 2$. This fits well with the philosophy in the work of Nick Katz according to which the "large finite field limit" should reflect behavior similar to what is expected over number fields. We also emphasize, as was noted above, that in Theorem 3.8 we prove (for $p \neq 2$) that our Möbius correction factor is usually nontrivial over large finite fields as f varies in suitable algebraic families. At the end of §6 we address analogues of our asymptotic and non-triviality results for $p = 2$.

We now give a brief outline of the paper. In §2 we formulate our corrected higher-genus conjecture of Bateman–Horn type. In §3 we use Möbius periodicity to study three aspects of our correction factor in odd characteristic: (i) its periodicity, (ii) its asymptotic structure for a single $f \in A[T^p]$ (considered over κ'/κ with $[\kappa':\kappa] \to \infty$) as well as "on average" for f varying in suitable families, and (iii) its non-triviality "on average" for varying f over large finite fields. The case of characteristic 2 is treated in §4–§6; the main difficulty here is to find and work with suitable 2-adic liftings (which we analyze via formal and rigid geometry). The appendix addresses numerical testing of our modified conjecture in §2.

Notation and Terminology. Our notation and terminology is largely as in [5]. For a nonarchimedean place v on a global field (for us, this will always be a function field over a finite field), $\mathrm{N}v$ denotes the size of its residue field. We will also use the symbol N in notation for the size of other residue rings or for a norm map between certain rings. The context should make clear the type of norm that is meant.

If $R \to R'$ is a map of rings and M is an R-module (or R-algebra) then $M_{R'}$ denotes $R' \otimes_R M$.

2. Higher-genus conjectures

Let k be a perfect field with characteristic $p > 0$ and let $C = \operatorname{Spec} A$ be a smooth affine geometrically connected curve over k with one geometric point ξ at infinity. Let $K = k(C)$, and let $f \in A[T^p]$ be squarefree in $K[T]$ and primitive with respect to A. Also assume $\deg_T f > 0$. We let g denote the genus of the smooth compactification $\overline{C}$ of C, and we write $\underline{V}_d$ to denote the affine space associated to the vector space $V_d = L(d \cdot \xi)$ for $d \in \mathbf{Z}$. For $d \geq 0$, let $V_d^0 = V_d - V_{d-1}$ and let $\underline{V}_d^0 = \underline{V}_d - \underline{V}_{d-1}$.

Lemma 2.1. *For $d \geq 2g$ there is a nonempty Zariski-open subset U_d in $\underline{V}_d^0$ such that for all perfect extensions k' of k, $U_d(k')$ is the set of $a \in \underline{V}_d^0(k') \subseteq k' \otimes_k A$ such that $f(a)$ is squarefree in $k' \otimes_k A$.*

If k is infinite or if k is finite and f has no local obstructions (i.e., the specialization $f_c \in \kappa(c)[T]$ is a nonzero function on $\kappa(c)$ for all $c \in C$), then $U_d(k)$ is nonempty for large d (with largeness only depending on the total degree $\deg_{u,T} f$ and the genus g).

Proof. Assume that $d \geq 2g$ and define U_d to be the Zariski-open complement of the union of the loci defined by the conditions "$a(c) = t$" on points a of the hyperplane complement $\underline{V}_d^0$, where $x = (c,t)$ ranges over points in the finite branch scheme B for the generically étale projection from $Z_f \subseteq C \times \mathbf{A}_k^1$ to $\mathbf{A}_k^1$, where Z_f is the zero scheme of $f \in A[T]$. (See [5, Lemma 2.2] for a proof that $Z_f \to \mathbf{A}_k^1$ is generically étale on Z_f.) To make explicit that the condition "$a(c) = t$" on a is an algebraic condition on $\underline{V}_d^0$, we use the norm-polynomial function $P_{x,d}(a) = \mathrm{N}_{k(x)/k}(a(c) - t)$ on $\underline{V}_d$. More precisely, the construction

$$P_{x,d}(a) = \mathrm{N}_{k' \otimes_k k(x)/k'}(a(c) - t) \in k'$$

for any k-algebra k' and $a \in k' \otimes_k V_d \subseteq k' \otimes_k A$ defines an algebraic function $\underline{V}_d \to \mathbf{A}^1_k$ whose vanishing locus away from the hyperplane $\underline{V}_{d-1}$ defines the condition "$a(c) = t$"; $\underline{V}_{d-1}$ is a hyperplane in $\underline{V}_d$ because $d \geq 2g$. Thus, we may take U_d to be the intersection of the nonvanishing loci of the $P_{x,d}$'s on $\underline{V}_d^0$ for $x \in B$. This is not empty because for each $x \in B$ the Riemann–Roch theorem ensures that $P_{x,d} \neq 0$ for $d \geq 2g$. If $B = \emptyset$ then we understand U_d to be $\underline{V}_d$. By [5, Theorem 2.5], U_d has the desired interpretation for its points with values in perfect extensions of k.

Since U_d is a nonempty open in an affine space over k for $d \geq 2g$, it has k-rational points when k is infinite. If k is finite and f has no local obstructions, to show $U_d(k)$ is nonempty (provided d is large enough, only depending on $\deg_{u,T} f$ and g) we use the one-variable case of a general theorem of Poonen [13, Thm 8.1] concerning squarefree specializations of squarefree polynomials in several variables over function fields of curves over finite fields. To formulate Poonen's result in our situation, for each closed point $c \in C$ define

$$n_c = \#\{\alpha \in \mathscr{O}_{C,c}/\mathfrak{m}_c^2 \mid f(\alpha) = 0\}.$$

Since f has no local obstructions, $n_c < \mathrm{N}(c)^2$ for all c. Poonen's theorem says

$$\lim_{n\to\infty} \frac{\#\{a \in A \mid -\mathrm{ord}_\xi(a) \leq d,\ \ f(a) \text{ squarefree}\}}{(q-1)q^{d-g}} = \prod_c \left(1 - \frac{n_c}{\mathrm{N}(c)^2}\right),$$

with the infinite product absolutely convergent, and in particular nonzero (the local factors are nonzero and Poonen shows $n_c = O(1)$ as $\mathrm{N}(c) \to \infty$). Letting $P > 0$ denote this infinite product, by subtracting consecutive terms in the limit we get

$$\lim_{d\to\infty} \frac{\#\{a \in A \mid -\mathrm{ord}_\xi(a) = d,\ \ f(a) \text{ squarefree}\}}{(q-1)q^{d-g}} = \left(1 - \frac{1}{q}\right) P > 0.$$

The numerator is $\#U_d(k)$, so we get the desired result for ineffective large d.

By [5, Thm. 2.5], the condition that $f(a)$ is squarefree only depends on $a \bmod I$, where $I = I_f$ is defined as above Theorem 1.1. The preceding limit calculation shows that this collection of congruence classes modulo I is not empty, and by the Riemann–Roch theorem each such congruence class admits a representative with any desired large pole order at ξ with largeness only depending on the genus and $\dim_\kappa(A/I)$. Hence, we just have to bound this dimension in terms of g and $\deg_{u,T} f$. By construction, $\dim_\kappa(A/I)$ is bounded above by the length of the branch scheme for the generically étale projection from $Z = Z_f \subseteq C \times \mathbf{A}^1$ to $\mathbf{A}^1$, so it is enough to bound the length of the branch scheme in terms of g and $\deg_{u,T} f$. If we let $D : A \to A$ be a nonzero k-linear derivation that has zero locus on C disjoint from the zeros of I then the branch scheme is contained in the overlap scheme $Z_f \cap Z_{Df}$ that is finite. By intersection theory on $\overline{C} \times \mathbf{P}^1$, the length of this overlap is bounded above in terms of $\deg_{u,T} f$ and the degree of the zero-scheme $Z(D)$ of D. By choosing D appropriately, we may bound $\deg(Z(D))$ in terms of the genus. ■

Now assume that $k = \kappa$ is finite, and we shall formulate a conjecture over A that is analogous to the one given in [3, Conj. 6.2] in the case $A = \kappa[u]$. The reasonableness of the conjecture will rest on the Möbius periodicity theorems in [5] in odd characteristic and the variants proved later in this paper (in §6) for $p = 2$. The conjecture provides a natural context for why such Möbius periodicity is useful and interesting. Another context is the parity problem in sieve theory, as we explained in [5, §1].

Pick $f \in A[T]$ that is squarefree in $K[T]$ and has no local obstructions; that is, for each place v of K distinct from the point ξ at infinity, we assume that the number

$$\omega_f(v) := \{\overline{a} \in \kappa(v) \mid f(\overline{a}) = 0\}$$

of roots of f in $\kappa(v)$ is strictly smaller than $\mathrm{N}v = \#\kappa(v)$. Define the infinite product

$$C_A(f) = \frac{1}{\mathrm{Res}(A)} \prod_{v \neq \xi} \frac{1 - \omega_f(v)/\,\mathrm{N}v}{1 - 1/\,\mathrm{N}v}, \tag{2.1}$$

where $\mathrm{Res}(A)$ is the residue at $s = 1$ for the zeta-function ζ_A of $\mathrm{Spec}\, A$ and the product runs over the places of K other than the unique point at infinity ξ for $\mathrm{Spec}\, A$. The product over v in (2.1) is generally only conditionally convergent, so it is understood to be an iterated product $\prod_{n \geq 1} \prod_{\mathrm{N}v = n}(\cdot)$ running over v according to increasing values of $\mathrm{N}v$. Since ξ is κ-rational, $\zeta_A(s) = L(q^{-s})/(1 - q \cdot q^{-s})$ with $L(t)$ a polynomial and q the size of the constant field κ of A. Obviously $\mathrm{Res}(A) = L(1/q)/\log q = h_{\overline{C}}/q^g \log q$, where $h_{\overline{C}} = \#\mathrm{Pic}^0_{\overline{C}/\kappa}(\kappa)$.

Assume $p \neq 2$ and $f \in A[T^p]$. For any nonzero ideal J of A, let

$$\Lambda_{A,J}(f; n) = 1 - \frac{\sum\limits_{\deg a = n, (f(a), J) = 1} \mu(f(a))}{\sum\limits_{\deg a = n, (f(a), J) = 1} |\mu(f(a))|}. \tag{2.2}$$

As we saw in the Introduction, the denominator of $\Lambda_{A,J}(f; n)$ is nonzero for $n \gg 0$, with largeness that only depends on $\deg_{u,T} f$, the genus g, and $\dim_\kappa(A/J)$. We will generally restrict attention to the case when J is a nonzero multiple of the radical ideal $I = I_f$ whose zero locus on C is the image in C of the finite branch scheme B for the generically étale projection $Z_f \to \mathbf{A}^1_\kappa$. Since $p \neq 2$, Theorem 1.1 (proved in §3) tells us that for any nonzero multiple J of I and for sufficiently large n (only depending on $\deg_{u,T} f$, the genus, and $\dim_\kappa(A/J)$) the function $\Lambda_{A,J}(f; n)$ is periodic in n with period 1, 2, or 4; the more precise formulation in Theorem 3.1 also gives that the periodic sequence of values of $\Lambda_{A,J}(f; n)$ for large n is *independent* of J.

Now consider $p = 2$ with $f \in A[T^4]$. Theorem 6.13 gives similar periodicity assertions for the analogue of (2.2) when J is taken to be any nonzero multiple of the radical of a certain nonzero (typically non-radical) ideal $I_{f,\kappa} \subseteq A$ replacing the role of the ideal I in the case of odd characteristic. Moreover, for any finite extension κ'/κ the ideal $\kappa' \otimes_\kappa I_{f,\kappa} \subseteq \kappa' \otimes_\kappa A$ is a multiple of $I_{f,\kappa'}$. (The definition of $I_{f,\kappa}$ is given in terms of the mod-2 reductions of certain radical characteristic-0 ideals constructed on 2-adic lifts of C; see Definition 6.4.) The finite zero-scheme of $I_{f,\kappa}$ on C has degree that is bounded in a manner only depending on $\deg_{u,T} f$ and the genus, not on κ. By Corollary 6.7, if we write $f = h(T^2)$ with $h \in A[T^2]$ then in the "generic" case that f has squarefree leading T-coefficient in A (*e.g.*, f is monic in T) the radical of $I_{f,\kappa}$ is equal to the radical ideal I_h whose zero locus on C is the image in C of the finite branch scheme of the generically étale projection $Z_h \to \mathbf{A}^1_\kappa$. Thus, $\Lambda_{A,I_{f,\kappa}}(f; n) = \Lambda_{A,I_h}(f; n)$ for such f.

To simplify notation, we shall write $\Lambda_A(f; n)$ for large n to denote the common periodic function $\Lambda_{A,J}(f; n)$ for large n and any nonzero multiple J of $I = I_f$ (resp. of $\mathrm{Rad}(I_{f,\kappa})$) when $p \neq 2$ (resp. $p = 2$); if we are interested in uniform largeness statements as we vary f or the finite base field then we shall take $J = I$ (resp. $J = \mathrm{Rad}(I_{f,\kappa})$). Often we will only be interested in considering sufficiently large n, so the use of the notation $\Lambda_A(f; n)$ (suppressing mention of J) will not create confusion.

For nonzero $a \in A$, set $\mathrm{N}(a) := \#(A/(a)) = q^{\deg a}$ with $q = \#\kappa$.

Conjecture 2.2. *Pick $f(T) \in A[T]$. Assume the following two conditions:*
1) *$f(T)$ is irreducible in $K[T]$,*
2) *$f(T)$ has no local obstructions.*

Let $\pi_f(n) = \#\{a \in A : \deg a = n, (f(a)) \text{ is prime}\}$. If f is separable over K then as $n \to \infty$,

$$\pi_f(n) \overset{?}{\sim} C_A(f) \sum_{\deg a = n} \frac{1}{\log(\mathrm{N}(f(a)))}. \tag{2.3}$$

If f is inseparable over K, with $f(T) \in A[T^4]$ if $p = 2$, then as $n \to \infty$,

$$\pi_f(n) \overset{?}{\sim} \Lambda_A(f; n) C_A(f) \sum_{\deg a = n} \frac{1}{\log(\mathrm{N}(f(a)))}. \tag{2.4}$$

Remark 2.3. One can give an alternative conjecture that treats the separable and inseparable cases on an equal footing and is equivalent to Conjecture 2.2 under a reasonable but unproved "randomness" hypothesis on $\mu(f(a))$ for K-separable $f \in A[T]$. Such an alternative conjecture is stated in [3, Rem. 6.3] for the case of genus 0, and its formulation carries over to any genus in a straightforward manner. We also note that although

$$\sum_{\deg a = n} \frac{1}{\log(\mathrm{N}(f(a)))} \sim \frac{(q-1)q^{n-g}}{n(\log q)\deg_T f} \tag{2.5}$$

as $n \to \infty$ (by Riemann–Roch), where g is the genus of the function field K/κ, we do not use this asymptotic estimate in numerical examples in the appendix because it gives poor accuracy for n in the range that can be used on a computer.

For $A = \kappa[u]$, Conjecture 2.2 for inseparable f is illustrated by numerical examples in [3, §6]. In the appendix we illustrate Conjecture 2.2 in examples with genera 1 and 2. This numerical data supports the use of the eventually periodic sequence $\Lambda_A(f; n)$ as a correction factor in (2.4). If 0 is in the periodic sequence of values of $\Lambda_A(f; n)$ for large n then the interpretation of (2.4) is that $\pi_f(n) = 0$ when $\Lambda_A(f; n) = 0$ and $n \gg 0$; this particular instance of (2.4) is easy to prove by unwinding the definition of $\Lambda_A(f; n)$, as we did in [3, §6] for $A = \kappa[u]$.

3. Möbius periodicity and asymptotics in odd characteristic

The following result will be essential for our later formulation of satisfactory asymptotic questions as we increase the constant field and consider the typical structure of averages of $\mu(f(a))$'s for f varying in certain families of polynomials.

Theorem 3.1. *Assume $p \neq 2$ and let $f \in A[T^p]$ be squarefree in $K[T]$ and primitive with respect to A. Let the nonzero radical ideal $I = I_f \subseteq A$ have zero locus on $C = \operatorname{Spec} A$ equal to the image in C of the finite branch scheme B of the generically étale projection $Z_f = \operatorname{Spec}(A[T]/(f)) \to \mathbf{A}^1_\kappa$. Assume moreover that f has no local obstructions: for all $c \in C = \operatorname{Spec}(A)$, the nonzero specialization $f_c \in \kappa(c)[T]$ does not vanish as a function on $\kappa(c)$. Let J be any nonzero multiple of I.*

There exists an n_0 only depending on the genus g of K/κ, the total degree $\deg_{u,T} f$, and $\dim_\kappa(A/J)$ such that for all $n \geq n_0$ there exists $a \in A$ with $\deg a = n$ such that $(f(a), J) = 1$

and $\mu(f(a)) \neq 0$. Moreover, by choosing this n_0 suitably we can arrange that

$$n \mapsto \frac{\sum_{\deg a = n, (f(a), J) = 1} \mu(f(a))}{\sum_{\deg a = n, (f(a), J) = 1} |\mu(f(a))|} \tag{3.1}$$

is periodic in $n \geq n_0$ with period dividing 4. If -1 is a square in $\kappa^\times$ or $\deg_T f$ is even then the period divides 2.

For any two nonzero multiples J_1 and J_2 of I, the functions defined by (3.1) for $J = J_1$ and $J = J_2$ are equal for $n \geq n_0'$ with n_0' determined by $\deg_{u,T} f$, the genus g, and the $\dim_\kappa(A/J_i)$'s.

Proof. Fix a κ-basis $\underline{\varepsilon} = \{\varepsilon_i\}_{i \geq 1}$ for A with $-\mathrm{ord}_\xi(\varepsilon_i)$ strictly increasing in i, and for any $n \geq 2g$ and $a \in V_n^0 = L(n\xi) - L((n-1)\xi)$ define $\mathrm{lead}(a) \in \kappa^\times$ to be the ε_{n+1-g}-coefficient in the expansion of a with respect to the basis $\underline{\varepsilon}$. (This is analogous to a leading coefficient in the sense of a Laurent expansion for a at ξ.) We view the vector space A/I as an affine space $\mathbf{V}$ over $\mathrm{Spec}\,\kappa$, and for each $x = (u_x, t_x) \in B \subseteq C \times \mathbf{A}^1$ we view the norm operations $P_x : h \mapsto \mathrm{N}_{\kappa(x)/\kappa}(h(u_x) - t_x)$ as algebraic functions on $\mathbf{V}$ in the evident manner. Define an algebraic function $\mathcal{L}$ on $\mathbf{V}$ by

$$\mathcal{L} = \prod_{x \in B} P_x^{e_x} = (h \mapsto \mathrm{N}_{B/\kappa}(h - T)), \tag{3.2}$$

with $e_x = \ell(\mathscr{O}_{B,x})$. By [5, Thm. 1.4, Thm. 3.1, (3.14)] we get that for all sufficiently large n (only depending on g and $\deg_{u,T} f$) and all $a \in A$ with $\deg a = n$,

$$\begin{aligned} \mu(f(a)) &= (-1)^{\dim(A/(\mathrm{lead}\, f)) + n \deg_T f} \chi(b_n (\mathrm{lead}\, a)^{e_n}) \chi(\mathcal{L}(a \bmod I)) \\ &= c_0 c_1^n \chi(b_n (\mathrm{lead}\, a)^{e_n}) \chi(\mathcal{L}(a \bmod I)), \end{aligned} \tag{3.3}$$

where $c_0 = (-1)^{\dim(A/(\mathrm{lead}\, f))}$, $c_1 = (-1)^{\deg_T f}$, χ is the quadratic character of $\kappa^\times$ ($\chi(0) = 0$), and the elements $e_n \in \mathbf{Z}$ and $b_n \in \kappa^\times$ depend on the choice of $\underline{\varepsilon}$. By [5, Thm. 3.6], $e_n \bmod 2$ is independent of $\underline{\varepsilon}$ and only depends on $n \bmod 2$, and if e_n is even then $b_n \bmod (\kappa^\times)^2$ is independent of $\underline{\varepsilon}$ and only depends on $n \bmod 4$ (and only depends on $n \bmod 2$ if moreover -1 is a square in $\kappa^\times$ or $\deg_T f$ is even).

Let $\mathcal{R}_J \subseteq A$ be a set of representatives for A/J; this set may be chosen so that each of its members has degree (i.e., pole order at ξ) bounded above in terms of g and $\dim_\kappa(A/J)$. We restrict attention to n large as above and also larger than the degrees of the elements in $\mathcal{R}_J$. Hence, for $a \in V_n^0$ with representative $R_a \in \mathcal{R}_J$ for the residue class of a in A/J we have $a - R_a \in J \cap V_n^0$ and $\mathrm{lead}(a) = \mathrm{lead}(a - R_a)$.

For each $R \in \mathcal{R}_J$, as a runs over $(R + J) \cap V_n^0 = R + (J \cap V_n^0)$ we see that $a - R$ runs through $J \cap V_n^0$ with each possible value of $\mathrm{lead}(a) \in \kappa^\times$ realized equally often. Also, the condition $(f(a), J) = 1$ is equivalent to the condition $(f(R_a), J) = 1$ since $a \equiv R_a \bmod J$. Thus, upon substituting (3.3) into the numerator and denominator on the right side of (3.1), the numerator of (3.1) equals

$$c_0 c_1^n \sum_{Q \in J \cap V_n^0} \sum_{\substack{\chi(\mathcal{L}(r \bmod I)) \neq 0 \\ f(r) \in (A/J)^\times}} \chi(b_n (\mathrm{lead}\, Q)^{e_n}) \chi(\mathcal{L}(r \bmod I)),$$

where the inner sum runs over r (which is ranging through A/J). This double sum equals the product

$$\chi(b_n) \sum_{\substack{\chi(\mathcal{L}(r \bmod I))\neq 0 \\ f(r)\in(A/J)^\times}} \chi(\mathcal{L}(r \bmod I)) \cdot \sum_{Q\in J\cap V_n^0} \chi(\operatorname{lead} Q)^{e_n}. \tag{3.4}$$

We fix the congruence class of n modulo 4 (resp. modulo 2 if -1 is a square in κ or if $\deg_T f$ is even), so by taking n to be large as above we may suppose that the parity of e_n is fixed.

Case 1: Suppose e_n is even. The terms in the second sum in (3.4) all equal 1, so (3.1) equals

$$c_0 c_1^n \chi(b_n) \cdot \frac{\sum_{\chi(\mathcal{L}(r \bmod I))\neq 0, f(r)\in(A/J)^\times} \chi(\mathcal{L}(r \bmod I))}{\#\{r \in A/J \mid \chi(\mathcal{L}(r \bmod I)) \neq 0, f(r) \in (A/J)^\times\}}, \tag{3.5}$$

where the dependence on n only occurs in $c_1^n\chi(b_n)$. By the evenness of e_n, the sign $\chi(b_n)$ is independent of $\underline{\varepsilon}$ and only depends on $n \bmod 4$ (or $n \bmod 2$ if -1 is a square in $\kappa^\times$ or $\deg_T f$ is even), so we have the desired dependence on $n \bmod 4$ for the large n that we are considering (and the dependence is on $n \bmod 2$ if -1 is a square in $\kappa^\times$ or $\deg_T f$ is even). To establish the independence of the choice of J in this case (for large n with e_n even), we show that (3.5) is independent of this choice. Write the multiple J of I in the form $J = J'J''$ with J' having the same prime factors as I and $(I, J'') = 1$. The ring A/J decomposes as a product $(A/J') \times (A/J'')$. Under this decomposition, write r as (r', r''). Then $\chi(\mathcal{L}(r \bmod I)) = \chi(\mathcal{L}(r' \bmod I))$, and $f(r) \in (A/J)^\times$ if and only if $f(r') \in (A/J')^\times$ and $f(r'') \in (A/J'')^\times$. Therefore the numerator and (nonzero) denominator sums in (3.5) are each given by multiplying the product

$$[I : J'] \cdot \#\{r'' \in A/J'' \mid f(r'') \in (A/J'')^\times\} \tag{3.6}$$

against the numerator and denominator sums in (3.5) with I in the role of J. The common factor (3.6) cancels out in the ratio.

Case 2: Suppose e_n is odd. We claim that (3.1) vanishes, or more specifically that the second sum in (3.4) vanishes. As we noted above, each element of $\kappa^\times$ arises equally often in the form $\operatorname{lead} Q$ as Q ranges over $J \cap V_n^0$, so this second sum is a multiple of the character sum for χ over $\kappa^\times$. This character sum vanishes since χ is nontrivial. ∎

Remark 3.2. With notation and hypotheses as above, consider the set of nonzero ideals J of A such that for $a, a' \in A$ with $\deg(a), \deg(a') \gg 0$,

$$a \equiv a' \bmod J,\ \frac{a}{a'} \in (K_\xi^\times)^2,\ \deg(a) \equiv \deg(a') \bmod 4 \Rightarrow \mu(f(a)) = \mu(f(a')). \tag{3.7}$$

One such ideal is $J = I$, by [5, Thm. 1.2]. It is obvious that if J_1 and J_2 are two such ideals then (J_1, J_2) is another such ideal. Hence, there is a minimal such ideal I_0 and $I_0|I$. (For example, in the notation of [3, Def. 3.4, Thm. 4.8], for $A = \kappa[u]$ we have $I = (M_f^{\mathrm{geom}})$ and $I_0 = (M_{f,\kappa}^{\min})$.) By (3.3), the nonzero ideals J that "work" in (3.7) are precisely those such that the set-theoretic function $a \mapsto \chi(\mathcal{L}(a \bmod I))$ only depends on $a \bmod J$. Thus, Theorem 3.1 remains true (with the same proof) if we replace I with I_0 throughout. However, the formation of I is compatible with finite extension on κ whereas the formation of I_0 generally is not.

For $p \neq 2$, we define $\overline{\mu}_{f,\kappa}(n)$ to be (3.1) for large n with $J = I$ (though any nonzero multiple J of I gives a function with the same tail). The *periodic part* of this function is its periodic sequence of values $\overline{\mu}_{f,\kappa}(n)$ for large n, and this largeness only depends on the genus g and the total degree $\deg_{u,T} f$.

Corollary 3.3. *Any nonzero terms in the periodic part of $\overline{\mu}_{f,\kappa}$ are equal up to sign.*

Proof. By the proof of Theorem 3.1, a nonzero term occurs for $n \gg 0$ only when e_n is even. An alternative expression for $\overline{\mu}_{f,\kappa}(n)$ in this case is (3.5) with $J = I$, in which changing $n \bmod 4$ only affects the term $c_1^n \chi(b_n)$ with $c_1 = \pm 1$. This is at most a sign change. ■

Corollary 3.4. *Suppose the periodic part of $\overline{\mu}_{f,\kappa}$ is not identically* 0 *but contains* 0*. Then* 0 *occurs in alternate terms of the periodic part and for any finite extension κ'/κ the periodic part of $\overline{\mu}_{f,\kappa'}$ vanishes at any large n where the periodic part of $\overline{\mu}_{f,\kappa}$ vanishes.*

Proof. If n is large and e_n is odd then $\overline{\mu}_{f,\kappa}(n) = 0$ by Case 2 in the proof of Theorem 3.1. If instead e_n is even then $\overline{\mu}_{f,\kappa}(n) = 0$ if and only if the numerator sum in (3.5) vanishes, but this vanishing is independent of n so such vanishing means that $\overline{\mu}_{f,\kappa}(n) = 0$ for all large n. Therefore, if the periodic part of $\overline{\mu}_{f,\kappa}$ contains 0 but is not identically 0 then it vanishes at the nth term (for large n) if and only if e_n is odd. Since the parity of e_n is determined by $n \bmod 2$ for large n [5, Thm. 3.6], the terms in the periodic part are alternately zero and nonzero in such cases.

If κ is replaced by a finite extension κ' then by construction (see [5, (3.14)]) $e_n \bmod 2$ does not change, so for $n \gg 0$ (with largeness only depending on g and $\deg_{u,T} f$) the Möbius average in degree n over κ' vanishes if the Möbius average in degree n over κ vanishes. ■

Since $e_n \bmod 2$ only depends on $n \bmod 2$ for large n, the proofs of Corollaries 3.3 and 3.4 show that in the periodic part of the sequence $\{\Lambda_A(f;n)\}_{n\gg 0}$ at most two values other than 1 can occur, and that if two such values do occur then their average is 1 (*e.g.*, the empirical pattern for $\Lambda_A(f;n)$ in Example A.6 that we expect to be the periodic part is $8/9, 1, 10/9, 1$). Corollary 3.4 says that any 1's in $\{\Lambda_A(f;n)\}_{n\gg 0}$ show up in alternate terms if the tail of this sequence is not identically 1 and that in such cases an extension of the constant field will not change any such 1's into other numbers. For example, in [3, Ex. 6.6] we calculated that $\{\Lambda_{\mathbf{F}_3[u]}(T^3 + u; n)\}_{n\gg 0}$ has periodic part 1,2,1,0 and $\{\Lambda_{\mathbf{F}_9[u]}(T^3 + u; n)\}_{n\gg 0}$ has periodic part 1,0,1,0. The following corollary and Case 1 in its proof yield Theorem 1.1.

Corollary 3.5. *Fix κ and $f \in A[T^p]$ as above with $p \neq 2$, and choose $c \in \{0,1,2,3\}$. Let $\overline{\mu}_{f,\kappa,c} = \overline{\mu}_{f,\kappa}(n)$ for large n with $n \equiv c \bmod 4$. For finite extensions κ'/κ, the Möbius average $\overline{\mu}_{f,\kappa',c}$ either tends to* 0 *as $[\kappa' : \kappa] \to \infty$ or lies in $\{\pm 1\}$ for all κ' with value depending only on the parity of $[\kappa' : \kappa]$.*

Proof. Fix a κ-basis $\underline{\varepsilon}$ of A as in the proof of Theorem 3.1. The exponents $\{e_x\}_{x\in B}$ in (3.2) are unchanged when κ is replaced by a finite extension, although extending κ may decompose a given point x into several x_i's, where $e_{x_i} = e_x$. Likewise, by construction, the exponent e_n in (3.3) that only matters modulo 2 is unaffected by replacing κ with a finite extension, and $e_n \bmod 2$ only depends on $n \bmod 2$ for large $n \equiv c \bmod 4$ (uniformly with respect to κ'/κ), so $e_n \bmod 2$ for such n is determined by c. Let $\sigma_c = (-1)^{e_n}$ for such n. There are three cases to consider (and each case is stable under finite extension on κ):

<u>Case 1</u>: Assume some e_x is odd (so in particular, the finite branch scheme B for the generically étale projection $Z_f \to \mathbf{A}^1_\kappa$ is nonempty). If $\sigma_c = -1$, so e_n is odd for large n in our fixed residue class $c \bmod 4$, then the Möbius average $\overline{\mu}_{f,\kappa',c}$ vanishes for all finite extensions κ'/κ (see the proof of Corollary 3.4). Assume now that $\sigma_c = 1$, so $\overline{\mu}_{f,\kappa,c}$ is given by (3.5) with $J = I$ and large n satisfying $n \equiv c \bmod 4$. We are going to show that, as κ

is replaced by finite extensions κ_m with degree m tending to ∞, the absolute value of (3.5) computed for A_{κ_m} with $J = I_{\kappa_m}$ is $O(q^{-m/2})$, where $q = \#\kappa$, so it tends to 0 as $m \to \infty$.

We interpret (3.5) in terms of point-counting on varieties to get an estimate on its size. We will ignore the factor $c_0 c_1^n \chi(b_n)$, which has absolute value 1 anyway. Let $\mathscr{A}$ be the ring scheme over κ corresponding to the finite κ-algebra A/I, and let $\mathscr{A}^\times$ be its algebraic unit group. Let $\mathfrak{L}$ be the nonzero algebraic function on $\mathscr{A}$ given by (3.2), and let U be the non-vanishing locus of $\mathfrak{L}$ on the preimage of $\mathscr{A}^\times$ under the evaluation mapping $f : \mathscr{A} \to \mathscr{A}$. Since f is primitive with respect to A, clearly $f^{-1}(U) \neq \emptyset$. Let V be the finite étale double cover of U given by the square root of $\mathfrak{L}|_U$. Since some e_x is odd, V is geometrically integral over κ.

We have $\dim U = \dim V (= \dim_\kappa(A/I))$. Call this common dimension d. Pick a finite extension κ_m/κ of degree m and consider (3.5) over κ_m. The denominator in (3.5) over κ_m is $\#U(\kappa_m)$ and the numerator in (3.5) over κ_m is $\#V(\kappa_m) - \#U(\kappa_m)$. The Lang–Weil estimate [11] may be applied to U and V since each is geometrically integral over κ, and the resulting estimate for each of $\#U(\kappa_m)$ and $\#V(\kappa_m)$ as $m \to \infty$ is the same: $q^{md} + O(q^{m(d-1/2)})$. Hence, $(\#V(\kappa_m) - \#U(\kappa_m))/\#U(\kappa_m) = O(q^{-m/2})$.

<u>Case 2</u>: If $\sigma_c = -1$ then the Möbius averages $\overline{\mu}_{f,\kappa',c}$ all vanish (as in Case 1).

<u>Case 3</u>: Assume all e_x's are even and $\sigma_c = 1$. Since the e_x's are even, for any $r \in A/I$ the value of $\chi(\mathfrak{L}(r))$ is 1 if it is nonzero. Therefore the numerator and denominator of (3.5) are equal, which means that for $n \equiv c \bmod 4$ with n large we have

$$\overline{\mu}_{f,\kappa,c} = \overline{\mu}_{f,\kappa}(n) = c_0 c_1^n \chi(b_n) = (-1)^{\dim A/(\mathrm{lead}\, f)}(-1)^{n \deg_T f}\chi(b_n) = \pm 1.$$

If κ is replaced by an odd degree extension κ' then the quadratic character of b_n in the extension is unchanged, so $\overline{\mu}_{f,\kappa',c} = \overline{\mu}_{f,\kappa,c}$. If κ is replaced by an even degree extension κ' then the quadratic character of b_n becomes 1, so

$$\overline{\mu}_{f,\kappa',c} = (-1)^{\dim(A/(\mathrm{lead}\, f)) + n \deg_T f} = (-1)^{\dim(A/(\mathrm{lead}\, f))} \cdot (-1)^{c \deg_T f}.$$

■

What does the proof of Corollary 3.5 say about the common value in $\{\Lambda_A(f;n)\}_{n \gg 0}$ indexed by large n in a fixed congruence class modulo 4 as κ is replaced by finite extensions κ' of large degree? (The largeness in n may be taken uniformly with respect to κ' since f is fixed.) Roughly speaking, the "stable" values that occur are 1 (in Case 2) or 0 and 2 (in Case 3). Any term in the periodic part (over κ') other than 0, 1, or 2 must arise through Case 1 and is replaced by numbers tending to 1 as $[\kappa' : \kappa] \to \infty$. In Case 2 there is trivially limiting behavior toward the value 1. Finally, by [5, Thm. 1.4], if a value of 0 or 2 occurs in the period then we are in Case 1 (rather than Case 3) if and only if the branch scheme B has odd length at some point (in particular, B is nonempty), in which case this term in the periodic part is again replaced by numbers tending to 1 as $[\kappa' : \kappa] \to \infty$. For example, $f = T^3 + u$ over $\kappa/\mathbf{F}_3$ has $\Lambda_{\kappa[u]}$-values 0 or 2 that arise in Case 3.

We now wish to address the behavior of the Möbius average function $\overline{\mu}_{f,\kappa,c}$ for typical f over large finite fields. There are two topics we shall consider: results saying that this average is small for all $c \in \{0, 1, 2, 3\}$, and results saying that it is not identically zero. To this end, define the *shape* of a nonzero polynomial $f = \sum \alpha_i T^{e_i} \in A[T]$ (with $\{e_i\}$ strictly increasing and all α_i nonzero) to be the data consisting of the e_i's and the pole-orders $\rho_i = -\mathrm{ord}_\xi(\alpha_i)$ at ξ. We shall be interested in studying all $f \in (\kappa' \otimes_\kappa A)[T^p]$ with a fixed shape such that f is primitive over $\kappa' \otimes_\kappa A$ with positive T-degree and is squarefree over $\kappa' \otimes_\kappa K = \mathrm{Frac}(\kappa' \otimes_\kappa A)$. Since $Z_f \to \mathbf{A}^1$ is generically étale, in a natural way this set of

f's for varying κ'/κ is identified with the set of κ'-points of a Zariski-open locus in an affine space over κ. This is what we shall call a *family* of f's (with varying constant field κ'/κ and a fixed curve C), provided that it is not empty. All members of a family have the same T-degree and the same total degree. We shall restrict our attention to those κ'/κ for which $\#\kappa'$ is strictly larger than the common T-degree of the fixed shape, so all such f have no local obstructions.

Our above work shows that there is a large n_0 determined by the genus and the common total degree of the members of the family so that for all κ' (large as above) and f we have: (i) the denominator in the definition of $\overline{\mu}_{f,\kappa'}(n)$ is nonzero for all $n \geq n_0$, (ii) the function $\overline{\mu}_{f,\kappa'}(n)$ is periodic in $n \geq n_0$ with period dividing 4. For each $c \in \{0, 1, 2, 3\}$ and κ' and f as above we define $\overline{\mu}_{f,\kappa',c}$ to be the common value $\overline{\mu}_{f,\kappa'}(n)$ for all $n \equiv c \bmod 4$ with $n \geq n_0$. It is reasonable to fix $c \in \{0, 1, 2, 3\}$ and $\varepsilon > 0$ and to ask how often $|\overline{\mu}_{f,\kappa',c}| < \varepsilon$ as f varies for fixed κ'/κ. And what happens to this proportion of f's as $[\kappa' : \kappa] \to \infty$? In terms of the associated correction factors $\lambda_{f,\kappa',c} = \Lambda_{\kappa'\otimes_\kappa A}(f; n) = 1 - \overline{\mu}_{f,\kappa',c}$ for $n \equiv c \bmod 4$ with $n \geq n_0$, it is equivalent to consider how often $|\lambda_{f,\kappa',c} - 1| < \varepsilon$ as f varies over κ' and to study how this proportion behaves as $[\kappa' : \kappa] \to \infty$.

If Z_f is étale over $\mathbf{A}^1$ for a generic member of the family then for generic f in the family the associated algebraic function $\mathfrak{L}$ as in (3.2) is identically 1 (there are no x's) and so the Möbius average $\overline{\mu}_{f,\kappa'}(n)$ in each large degree n is equal to either ± 1 or 0, the latter case being precisely the one in which e_n is odd. In the case of genus 0, [3, Ex. 4.15] explicitly describes all families whose generic member f has Z_f étale over $\mathbf{A}^1$. In general it seems hopeless to give an explicit description, though in any particular case it is easy to determine if the generic member f has zero-scheme $Z_f \subseteq C \times \mathbf{A}^1$ that is étale over $\mathbf{A}^1$ (and for "most" families one expects Z_f for generic f to not be étale over the affine line). The following result solves our asymptotic problem for all families aside from those for which $Z_f \to \mathbf{A}^1$ is étale for the generic member of the family:

Theorem 3.6. *Fix κ and A as above (with $p \neq 2$). Consider a (nonempty) family $\mathscr{F} \subseteq \{f = \sum_i \alpha_i T^{e_i} \mid \deg(\alpha_i) = \rho_i\}$ in the sense defined above such that $p|e_i$ for all i and such that the generic member f has Z_f not étale over $\mathbf{A}^1$. Assume that there exist i_0 and i_1 such that $L(\rho_{i_0} \cdot \xi - 3y_0)$ has codimension 3 in $L(\rho_{i_0} \cdot \xi)$ for some geometric point $y_0 \in C$ and $L(\rho_{i_1} \cdot \xi - y_1) \neq L(\rho_{i_1} \cdot \xi - 2y_1)$ for all geometric points $y_1 \in C$. (This is automatically satisfied with $i_1 = i_0$ if $\rho_{i_0} \geq 2g + 2$ for some i_0.)*

There exists a Zariski-dense open locus $\mathscr{F}^0$ in the family $\mathscr{F}$ so that for all finite extensions κ'/κ and $f \in \mathscr{F}^0(\kappa')$ the branch scheme of $Z_f \to \mathbf{A}^1$ contains a point with p-power multiplicity, so $\lambda_{f,\kappa'',c} \to 1$ as $[\kappa'' : \kappa'] \to \infty$ for all $c \in \{0, 1, 2, 3\}$. Moreover, after possibly shrinking $\mathscr{F}^0$, this convergence is uniform in the sense that for all $\varepsilon > 0$ and $[\kappa' : \kappa] \gg 0$, $|\lambda_{f,\kappa',c} - 1| < \varepsilon$ for all $c \in \{0, 1, 2, 3\}$ and all $f \in \mathscr{F}^0(\kappa')$; the largeness condition on $[\kappa' : \kappa]$ only depends on ε, g, and the e_i's and ρ_i's.

It follows immediately from this theorem and the Lang–Weil estimate that for each $\varepsilon > 0$ the proportion of κ'-points f in the family such that $|\lambda_{f,\kappa',c} - 1| < \varepsilon$ for all $c \in \{0, 1, 2, 3\}$ tends to 1 as $[\kappa' : \kappa] \to \infty$. In the case of genus 0, the hypothesis that Z_f is not étale over $\mathbf{A}^1$ for generic f in the family $\mathscr{F}$ is equivalent to the condition that $\rho_{i_0} \geq 2 = 2g + 2$ for some i_0. (See [3, Ex. 4.15] for a proof.) Hence, the hypotheses in Theorem 3.6 concerning i_0 and i_1 are redundant in the case of genus 0.

Proof. We proceed in five steps.

Step 1. We begin by relativizing some of our preceding considerations over fields. The family $\mathscr{F}$ is parameterized by the points of a geometrically irreducible κ-scheme S of finite type. Let $\mathscr{Z} \subseteq C \times \mathbf{A}^1_S$ be the zero scheme of the universal member of the family, so for each $s \in S$ the fiber $\mathscr{Z}_s$ is the zero scheme Z_{f_s} of $f_s = \sum \alpha_i(s) T^{e_i} \in (\kappa(s) \otimes_\kappa A)[T^p]$ in $C \times \mathbf{A}^1_{\kappa(s)}$. By the local flatness criterion, $\mathscr{Z}$ is S-flat. For each geometric point s of S, the polynomial f_s is primitive with respect to $\kappa(s) \otimes_\kappa A$ and is squarefree in $\mathrm{Frac}(\kappa(s) \otimes_\kappa A)[T]$ by our definition of "family", so by [5, §2] the projection $\mathscr{Z} \to \mathbf{A}^1_S$ is quasi-finite and flat with étale locus that is dense in each fiber $\mathscr{Z}_s$. Define the relative branch scheme $\mathscr{B} \subseteq \mathscr{Z}$ to be the zero scheme of the Fitting ideal of $\Omega^1_{\mathscr{Z}/\mathbf{A}^1_S}$. The formation of $\mathscr{B}$ is compatible with base change on S, so $\mathscr{B}$ is quasi-finite over S. By a calculation with the local flatness criterion we see that $\mathscr{B}$ is also S-flat.

Let $\eta \in S$ be the generic point. For a suitable open neighborhood S^0 of η, the restriction $\mathscr{B}^0 = \mathscr{B}|_{S^0}$ is finite and flat over S^0 and its schematic image in $C \times S^0$ has underlying reduced scheme Y that is also finite and flat over S^0. In particular, the formation of the ideal of Y in $C \times S^0$ commutes with base change on S^0. We cannot expect Y to be étale over S^0, but for each geometric point $s \in S^0$ the fiber Y_s is defined by an ideal in $\kappa(s) \otimes_\kappa A$ with radical equal to the ideal I_{f_s} whose zero locus on C_s is the image in C_s of the finite branch scheme for the projection from Z_{f_s} to $\mathbf{A}^1$. By hypothesis this latter projection is non-étale for generic $s \in S$, so $Y \neq \emptyset$.

Letting $\psi : Y \to S^0$ be the finite flat structure map, consider the (positive-rank) vector bundle $\mathscr{W} = \psi_*(\mathscr{O}_Y)$ on S^0. This is an $\mathscr{O}_{S^0}$-subalgebra of the pushfoward of the structure sheaf of $\mathscr{B}^0$, and so we can define an S^0-map $\mathscr{W} \to \mathbf{A}^1_{S^0}$ by the functorial rule

$$h \mapsto \mathrm{N}_{\mathscr{B}^0/S^0}(h - T).$$

On fibers over closed points s of S^0 (or more generally, points s valued in a perfect field) this recovers (3.2) except that h is taken modulo a possibly non-radical ideal whose zero scheme on the fiber C_s has degree equal to the constant rank of Y over S^0.

Step 2. We now reduce our problem to an assertion about the relative branch scheme $\mathscr{B}$, and we identify S^0 with a space of hyperplanes. Let $m > 0$ be maximal such that $p^m | e_i$ for all i and write $f = F(T^{p^m})$ for the universal point f of the family. Let $\mathscr{B}_F$ be the relative branch scheme associated to F. As is explained at the end of [5, §4], for each geometric point s of S there is a natural bijection $b \mapsto b'$ from the set of points of the fiber $\mathscr{B}_s$ to the set of points of the fiber $(\mathscr{B}_F)_s$, and the lengths of the artinian local rings at these points satisfy $\ell(\mathscr{O}_{\mathscr{B}_s,b}) = p^m \ell(\mathscr{O}_{(\mathscr{B}_F)_s,b'})$. Hence, if we can find an étale point on $(\mathscr{B}_F)_\eta$ then after shrinking S^0 we can arrange that for each finite extension κ'/κ and $s \in S^0(\kappa')$ the fiber $\mathscr{B}_s$ has odd length p^m at some point. The argument in Case 1 (especially with $\sigma_c = 1$) in the proof of Corollary 3.5 works uniformly across all fibers over $S^0(\kappa')$ for all κ'/κ because the O-constant in the Lang–Weil estimate is uniform in algebraic families, so to conclude the proof of the theorem it suffices to find an étale point on $(\mathscr{B}_F)_\eta$. In Step 1 we saw that $\mathscr{B}_\eta \neq \emptyset$, so $(\mathscr{B}_F)_\eta \neq \emptyset$..

Write $e_i = p^m e_i'$, so some e_i' is not divisible by p. Let $\mathbf{A}^N$ be the affine space with coordinates labelled by the finite set of elements $a_{ij} T^{e_i'}$ with $\{a_{ij}\}_j$ a κ-basis of $L(\rho_i \cdot \xi)$ for $i > 0$ and $\{a_{0j}\}_j \subseteq L(\rho_0 \cdot \xi)$ representing a κ-basis of the quotient space $L(\rho_0 \cdot \xi)/\kappa$ for $i = 0$. (Note that $e_0' = 0$, as otherwise all f's would be divisible by T^p and so would not be squarefree, contrary to the assumption that the family $\mathscr{F}$ is not empty.) There is a canonical map

$$\pi : C \times \mathbf{A}^1 \to \mathbf{A}^N$$

defined by $(c,t) \mapsto (a_{ij}(c)t^{e'_i})_{i,j}$. Since some e'_i is not divisible by p (necessarily this i must be positive), and any $L(\rho\cdot\xi)$ larger than κ cannot consist entirely of pth powers in $\kappa(C)$ (so $\kappa(C)$ is finite separable over $\kappa(h)$ for some $h \in L(\rho\cdot\xi) - \kappa$), the map π is generically étale onto its image. Let $U \subseteq C \times \mathbf{A}^1$ be the maximal open subscheme on which π is étale. It is clear that U contains $C' \times (\mathbf{A}^1 - \{0\})$ for a sufficiently small dense open $C' \subseteq C$, so the positive-dimensional irreducible components of $(C \times \mathbf{A}^1) - U$ are either of the form $\{c\} \times \mathbf{A}^1$ or $C \times \{0\}$. The restriction of π to each component of the first type is generically étale, and the restriction of π to $C \times \{0\}$ is either constant (if $L(\rho_0 \cdot \xi) = \kappa$) or generically étale. It therefore follows from Bertini's theorem [10, I, Thm. 6.3] that for a Zariski-dense open locus of affine hyperplanes H in $\mathbf{A}^N$ the pullback $\pi^{-1}(H)$ is a geometrically irreducible and smooth curve in $C \times \mathbf{A}^1$; in fact, for generic H the pullback $\pi^{-1}(H)$ meets U in a smooth curve and also has étale overlap with the smooth locus of each curve component of the complement of U in $C \times \mathbf{A}^1$. Since $e'_0 = 0$, these varying $\pi^{-1}(H)$'s are precisely the zero-schemes Z_F when we restrict our attention to the dense open locus of H's for which the corresponding polynomial $F(T^{p^m})$ is squarefree in $K[T]$ and primitive with respect to A. Thus, we may identify S^0 with an open subscheme of the space of H's, and for all H in this open locus the projection from $\pi^{-1}(H)$ to $\mathbf{A}^1$ is quasi-finite.

Step 3. By hypothesis there is a geometric point y_0 of C such that $L(\rho_{i_0} \cdot \xi - 3y_0)$ has codimension 3 in $L(\rho_{i_0} \cdot \xi)$ for some i_0. By semi-continuity, the same holds for any y_0 in a dense open $C^0 \subseteq C$. For any $(y_0, t_0) \in C^0 \times \mathbf{G}_m$ we can certainly find an affine hyperplane $H_0 \subseteq \mathbf{A}^N$ such that $\pi^{-1}(H_0) \cap C_{t_0}$ contains y_0 as an isolated point with length 2. (Such an H_0 can be found so that its defining equation has vanishing coefficients for coordinates away from those corresponding to a basis of $L(\rho_{i_0} \cdot \xi)$.) Also, for any $x_0 = (y_0, t_0) \in C^0 \times \mathbf{G}_m$ the tangent map $\mathrm{d}\pi(x_0)$ is injective on the line $\mathrm{T}_{y_0}(C_{t_0})$.

In general, for any affine hyperplane H_0 in $\mathbf{A}^N$ and any point $x_0 = (y_0, t_0) \in C^0 \times \mathbf{G}_m$, the condition that $\pi^{-1}(H_0) \cap C_{t_0}$ contains $x_0 = (y_0, t_0)$ as an isolated point with length greater than 1 is precisely the condition that $\pi^{-1}(H_0) \to C$ is quasi-finite at x_0 and the affine line $\mathrm{d}\pi(x_0)(\mathrm{T}_{y_0}(C_{t_0}))$ through $\pi(x_0)$ is contained in the affine hyperplane H_0 through $\pi(x_0)$. The condition of having length greater than 1 at x_0 is equivalent to the projection $\pi^{-1}(H_0) \to \mathbf{A}^1$ being non-étale at x_0 when $\pi^{-1}(H_0)$ is smooth at x_0 and quasi-finite over $\mathbf{A}^1$ at x_0. Also, if $\pi^{-1}(H_0) \cap C_{t_0}$ has length 2 at x_0 and $\pi^{-1}(H_0)$ is smooth at x_0 then the branch scheme for the projection from $\pi^{-1}(H_0)$ to $\mathbf{A}^1$ is étale at x_0 because $p \neq 2$. However, it is not *a priori* evident if the smoothness condition for $\pi^{-1}(H_0)$ at x_0 is a generic property when we require that $\pi^{-1}(H_0) \cap C_{t_0}$ has length > 1 at x_0, so we shall avoid imposing such a smoothness requirement in our study of such triples (H_0, t_0, x_0).

Step 4. Consider the incidence scheme Σ consisting of triples (H, t, x) with H an affine hyperplane in $\mathbf{A}^N$ and $x = (y, t) \in C^0 \times \mathbf{G}_m$ a point of $\pi^{-1}(H)$ lying over $t \in \mathbf{G}_m$ such that the map $\pi^{-1}(H) \to C$ is quasi-finite at x and H contains the tangent line $\mathrm{T}_y(C_t)$ (viewed as an affine line in $\mathbf{A}^N$ via $\mathrm{d}\pi(x)$). This incidence scheme makes sense because of openness of the quasi-finite locus for a morphism of finite type [7, IV$_3$, 13.1.4]. By the infinitesimal smoothness criterion we see that the projection $\Sigma \to C^0 \times \mathbf{G}_m$ is a smooth map. Each fiber of this map is a dense open in the space of affine hyperplanes in $\mathbf{A}^N$ containing a common line, so Σ is irreducible with dimension $2 + (N-2) = N$. The projection from Σ to the space of affine hyperplanes in $\mathbf{A}^N$ has quasi-finite generic fiber because for a generic choice of H the preimage $\pi^{-1}(H)$ is a smooth and geometrically irreducible curve in $C \times \mathbf{A}^1$ whose projection to $\mathbf{A}^1$ is quasi-finite and generically étale. But is this generic fiber in Σ perhaps empty? The fiber of Σ over a generic point H is not empty *provided* that the generic branch scheme $(\mathscr{B}_F)_\eta$ is not contained in the union of $C \times \{0\}$ and finitely many vertical slices of the form $\{c\} \times \mathbf{A}^1$ for closed points $c \in C$ (such as the points of $C - C^0$). Since $(\mathscr{B}_F)_\eta \neq \emptyset$, this problem for $(\mathscr{B}_F)_\eta$ is settled by:

Lemma 3.7. *No point of the η-finite generic branch scheme $(\mathscr{B}_F)_\eta$ is contained in $C \times \{0\}$ or in $\{c\} \times \mathbf{A}^1$ for any closed point $c \in C$.*

Proof. It is equivalent to work with the generic branch scheme $\mathscr{B}_\eta$ for the associated family of polynomials $f = F(T^{p^m}) = \sum \alpha_i T^{e_i}$ in T^p, in view of the definition of the bijection of branch schemes defined at the end of [5, §4], so let us now work with the latter branch scheme $\mathscr{B}_\eta$. The generic element of $L(\rho_0 \cdot \xi)$ has étale zero-scheme on C, so the generic branch scheme does not meet $C \times \{0\}$. Suppose instead that the generic branch scheme for the family of f's meets $\{c\} \times \mathbf{A}^1$ for some closed point c of C. Let $D : A \to A$ be a κ-derivation that induces a basis of the cotangent space at c. Working over an algebraic closure $\overline{\kappa(c)}$ of $\kappa(c)$, by the proof of [5, Thm. 2.5] the condition for $\{c\} \times \mathbf{A}^1$ to meet the generic branch scheme is that the polynomials $\sum_i \alpha_i(c) T^{e_i}$ and $\sum_i (D\alpha_i)(c) T^{e_i}$ are not relatively prime for any (α_i) lying in some dense open locus in the affine space $\prod_i L(\rho_i \cdot \xi)$ over $\mathrm{Spec}(\kappa)$.

To prove that this is impossible, note that we can choose the α_i's so that the tuple $(\alpha_i + b)_i$ is as generic as we wish for b ranging through a fixed subset of $\overline{\kappa(c)}$ with cardinality exceeding $\max_i e_i$. Since $D(\alpha_i + b) = D\alpha_i$, it follows that for such (α_i) each of the polynomials $\sum_i (\alpha_i(c) + b) T^{e_i}$ has a root in common with $\sum_i (D\alpha_i)(c) T^{e_i}$. The point c is not a zero of all $D\alpha_i$'s, because by hypothesis there exists $\alpha_{i_1} \in L(\rho_{i_1} \cdot \xi)$ with a simple zero at c (so $(D\alpha_{i_1})(c) \neq 0$). Hence, by the pigeonhole principle there exist distinct elements $b_1, b_2 \in \overline{\kappa(c)}$ such that the polynomials $\sum_i (\alpha_i(c) + b_1) T^{e_i}$ and $\sum_i (\alpha_i(c) + b_2) T^{e_i}$ have a common root for generic (α_i). This common root is also a root of the nonzero polynomial $\sum_i T^{e_i}$, so it lies in $\overline{\kappa(c)}$. It follows that for sufficiently generic choices of (α_i) the polynomials $\sum_i \alpha_i(c) T^{e_i}$ have a common root $r_0 \in \overline{\kappa(c)}$, and this is a contradiction since we can fix all α_i for $i > 0$ and add all but finitely many constants to α_0 without affecting genericity but certainly destroying the property of r_0 being a root (because $e_0 = 0$). ∎

Step 5. By Lemma 3.7, the generic point of Σ maps to the generic point of the space of affine hyperplanes in $\mathbf{A}^N$. In particular, the generic fiber $(\mathscr{B}_F)_\eta$ contains an étale point if and only if for the generic point (H_η, t_η, x_η) of Σ the point x_η is in the étale locus of the branch scheme for the quasi-finite and generically étale map $\pi^{-1}(H_\eta) \to \mathbf{A}^1$ (with $\pi^{-1}(H_\eta)$ a smooth and geometrically irreducible curve).

Consider the universal triple $(H^{\mathrm{univ}}, t^{\mathrm{univ}}, x^{\mathrm{univ}})$ over Σ. Let $\pi_\Sigma : C \times \mathbf{A}^1_\Sigma \to \mathbf{A}^N_\Sigma$ denote the base change of π, so $X := \pi_\Sigma^{-1}(H^{\mathrm{univ}}) \cap C_{t^{\mathrm{univ}}}$ is a subscheme of $C \times \Sigma$ that is quasi-finite over Σ along the section x^{univ} and the fiber-degree of $X \to \Sigma$ along this section is pointwise greater than 1. By the local flatness criterion, $X \to \Sigma$ is also flat along x^{univ}. Hence, by the structure theorem for quasi-finite separated maps [7, IV$_4$, 18.5.11], the fiber-degree of $X \to \Sigma$ at the generic point of the section x^{univ} is bounded above by the fiber-degree at any point of this section (and is bounded below by 2, since all of these fiber degrees exceed 1 by definition of Σ). But we have already noted that for any geometric point $x = (y, t) \in C^0 \times \mathbf{G}_m$ there is an affine hyperplane H in $\mathbf{A}^N$ such that $(H, t, x) \in \Sigma$ and $\pi^{-1}(H) \cap C_t$ has length 2 at x. Thus, the fiber-degree at the generic point of x^{univ} is equal to 2. Since $\pi_\Sigma^{-1}(H^{\mathrm{univ}})$ has smooth fiber over the generic point of Σ, the corresponding branch scheme is therefore étale at the generic point of the section x^{univ}. ■

Theorem 3.6 makes precise the sense in which (for $p \neq 2$) the case when some e_x is odd (in fact, a p-power) is the "generic" case. In such cases, when considering the statistics for $\mu_{\kappa'\otimes_\kappa A}(f(a'))$ for $a' \in \kappa' \otimes_\kappa A$ as $[\kappa' : \kappa] \to \infty$ we are in Case 1 in the proof of Corollary 3.5 (so as $[\kappa' : \kappa] \to \infty$ we have $\Lambda_{\kappa'\otimes_\kappa A} \to 1$ as a function on $\mathbf{Z}/4\mathbf{Z}$). The following result shows that for such f it often happens that for finite extensions κ'/κ with sufficiently divisible degree the function $\Lambda_{\kappa'\otimes_\kappa A} = 1 - \overline{\mu}_{f,\kappa',I_f}$ on $\mathbf{Z}/4\mathbf{Z}$ is *not* identically 1.

Theorem 3.8. *Consider a non-empty algebraic family of polynomials $\mathscr{F}$ satisfying the hypotheses as in Theorem* 3.6*. Assume that the generic member of the family has odd T-degree and that the highest T-degree in which the generic coefficient is not constant is odd. Also assume that the family is not of the form $\{\alpha_1 T^{p^r} + \alpha_0\}$ with $r > 0$.*

There is a Zariski-dense open locus $\mathscr{F}' \subseteq \mathscr{F}$ and a positive integer δ depending only on the total degree of the members of $\mathscr{F}$ so that for any finite extension κ'/κ, any $f \in \mathscr{F}'(\kappa')$, and any finite extension κ''/κ' with $[\kappa'' : \kappa']$ divisible by δ, the function $n \mapsto \overline{\mu}_{f,\kappa'',I_f}(n)$ on $\mathbf{Z}/4\mathbf{Z}$ for large n is not identically zero.

Proof. By Theorem 3.6 we can pass to a Zariski-dense open locus in the family to arrange that the branch scheme B_f for the projection $Z_f \to \mathbf{A}^1$ has a point with odd multiplicity (and in particular B_f is not empty). That is, we can restrict our attention to Case 1 of the proof of Corollary 3.5. For any positive integer m the generic member of the linear system $|m\cdot\xi|$ has nonempty étale divisor on C if $\dim |m\cdot\xi| > 0$, so it is a further Zariski-dense open condition to require that the nonconstant coefficient in f occurring in highest T-degree has divisor on C that is nonempty and étale (and so not divisible by p).

Hence, it follows from [5, Thm. 6.3] that for generic f in the family we have that e_n is even for all large n in some congruence class modulo 4 (where e_n is as in (3.3)). Arguing (and using notation) as in Case 1 of the proof of Corollary 3.5, it is therefore enough to prove that if such an f is a κ'-point of the family then $\#V(\kappa'') - \#U(\kappa'') \neq 0$ when $[\kappa'' : \kappa']$ is sufficiently divisible (independent of κ' and f); recall that V and U depend on f. We will prove this in five steps.

Step 1. We shall first reduce the problem to the non-vanishing of certain quadratic character sums. The non-empty branch scheme $B = B_f$ for $Z_f \to \mathbf{A}^1$ has degree equal to a common (positive) value for generic f, and depending just on this degree (and not f) we can make $[\kappa'' : \kappa']$ sufficiently divisible so that B_{red} is κ''-split for generic f. The image of $B(\kappa'')$ in $C_{\kappa''}$ therefore is a finite nonempty set $Q = Q_f$ of κ''-rational points, and for each point $c \in Q$ we let B_c be the set of points in the nonempty fiber of $B(\kappa'')$ over c. Letting $f_c \in \kappa''[T]$ be the *nonzero* specialization of $f \in A_{\kappa'}[T^p]$ at $c \in C(\kappa'')$, we may and do take $[\kappa'' : \kappa']$ to be sufficiently divisible so that f_c is κ''-split for all $c \in Q$ and $\#\kappa'' > \deg_T f$. In particular, if $a \in A_{\kappa''}$ is such that $f(a)$ is a unit modulo I_f then for all $x = (u_x, t_x) \in B(\kappa'')$ the reduction $\overline{a} \in (A_{\kappa'}/I_f)_{\kappa''}$ has u_x-component (with respect to the decomposition of this quotient algebra into a product of copies of κ'') satisfying $f_{u_x}(\overline{a}(u_x)) \neq 0$, so $a(u_x) - t_x \neq 0$.

Hence, we have the formula

$$|\#V(\kappa'') - \#U(\kappa'')| = \left| \prod_{c\in Q} \sum_{z\in\kappa'' : f_c(z)\neq 0} \chi_{\kappa''}(P_c(z)) \right|$$

in which $\chi_{\kappa''}$ is the quadratic character on $(\kappa'')^\times$ ($\chi_{\kappa''}(0) = 0$), $P_c(Z) = \prod_{x \in B_c}(Z - t_x)^{e_x} \in \kappa''[Z]$ is a polynomial of positive degree, and the condition "$f_c(z) \neq 0$" says exactly that $f(a) \bmod I_f$ has unit component at c for $a \in A_{\kappa''}$ representing the c-component value $z \in \kappa''$. Note that if $f_c(z) \neq 0$ then the point (c, z) does not lie on the zero locus of f in $C \times \mathbf{A}^1$ and so it does not lie on B. Hence, $z \neq t_x$ for all $x = (c, t_x) \in B_c$ and thus $P_c(z) \neq 0$ for such z.

If $x \in B(\kappa'')$ and e_x is even then the value $(\overline{a}(u_x) - t_x)^{e_x} \in \kappa''$ is a nonzero square for any $\overline{a} \in (A_{\kappa'}/I_f)_{\kappa''}$ such that $f_{u_x}(\overline{a}(u_x)) \neq 0$, and for each $c \in Q$ the polynomial f_c is nonzero somewhere on κ'' since $\#\kappa'' > \deg_T f$. To prove the nonvanishing of $\#V(\kappa'') - \#U(\kappa'')$ when $[\kappa'' : \kappa']$ is sufficiently divisible (independent of κ' and f), it therefore suffices to prove that each 1-variable quadratic character sum $\sum_{f_c(z) \neq 0} \chi_{\kappa''}(P_c(z))$ for $z \in \kappa''$ is nonzero when $[\kappa'' : \kappa']$ is sufficiently divisible (in a manner that is independent of $c \in Q$).

Step 2. The quadratic character sums $\sum_{f_c(z) \neq 0} \chi_{\kappa''}(P_c(z))$ are related to point-counting on certain hyperelliptic curves, so we now reformulate our problem in terms of such curves. For $c \in Q$, define $R_c(Z) \in \kappa''[Z]$ to be the monic product of linear terms $Z - t_x$ for $x \in B_c$ such that e_x is odd, so $R_c | P_c$ and P_c/R_c is a square in $\kappa''[Z]$. Thus, $\chi_{\kappa''}(P_c(z)) = \chi_{\kappa''}(R_c(z))$ whenever $f_c(z) \neq 0$, so we may and do replace P_c with R_c in the quadratic character sums. The case $R_c = 1$ is trivial (as then the quadratic character sum for c is a positive integer, since f_c is nonzero somewhere on κ''), so we now restrict attention to c such that the separable polynomial R_c has positive degree.

Since the degree of the nonzero $f_c(T) \in \kappa''[T]$ is bounded by $\deg_T f$, with enough divisibility for $[\kappa'' : \kappa']$ we can ensure that $f_c \in \kappa''[T]$ is split and R_c has square value (possibly zero) at each zero of f_c. Consider the smooth affine curve $\mathscr{X}_c = \{W^2 = R_c(Z)\}$ with hyperelliptic compactification $\overline{\mathscr{X}}_c$. The curve $\mathscr{X}_c$ has 1 or 2 geometric points at infinity, and by passing to a quadratic extension if necessary we can assume that such points are κ''-rational. We shall separate the problem into two cases, when the genus g_c of $\overline{\mathscr{X}}_c$ is positive or zero, and these two cases will be respectively treated via the Riemann Hypothesis and via specialization arguments. (For "most" families $\mathscr{F}$ and $f \in \mathscr{F}$ chosen generically we probably have that $g_c = 0$, and even $\deg R_c = 1$, for all c: distinct branch points of $Z_f \to \mathbf{A}^1$ should lie in distinct fibers. However, we do not wish to impose the *ad hoc* hypothesis on $\mathscr{F}$ that this is the case, and we do not know simple hypotheses on the structure of the family that are sufficient to ensure it.)

Step 3. We first take care of the easier case $g_c > 0$ (*i.e.*, $\deg R_c \geq 3$). Note that g_c is bounded above independently of f in our family. Since

$$\sum_{f_c(z) \neq 0} \chi_{\kappa''}(R_c(z)) = \#\mathscr{X}_c(\kappa'') - \#\kappa'' - \#\{z \in \kappa'' \mid f_c(z) = 0, R_c(z) \neq 0\},$$

it suffices to arrange that $\#\overline{\mathscr{X}}_c(\kappa'') - (\#\kappa'' + 1)$ avoids integral values between 0 and $1 + \deg_T f$. By the construction of Q_f, the geometrically connected and smooth proper curve $\overline{\mathscr{X}}_c$ can be descended to a finite extension of κ' with degree bounded independently of κ'/κ and the $f \in \mathscr{F}(\kappa')$ that we are considering. Thus, by considering the $2g_c$ Frobenius eigenvalues in $\mathbf{C}$ associated to such a descent of $\overline{\mathscr{X}}_c$ we are reduced to the following concrete problem: if $S \subseteq \mathbf{Z}$ is a finite subset, $q = p^e$ is a prime power ($e > 0$), and $\{\alpha_1, \dots, \alpha_m\}$ is a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-stable multiset of Weil q-integers in $\mathbf{C}$ with positive weights then for all sufficiently divisible r (depending only on m, p, and S) the rational integer $\sum_j \alpha_j^r$ does not lie in S.

Pick a rational prime $\ell \neq p$ so that $s \not\equiv m \bmod \ell$ for all $s \in S$ with $s \neq m$. Let $K = \mathbf{Q}(\alpha_1, \ldots, \alpha_m)$ and let λ be a prime of K over ℓ, so $\alpha_j^{\mathrm{N}(\lambda)-1} \equiv 1 \bmod \lambda$ for all j. Hence, taking r divisible by $\mathrm{N}(\lambda) - 1$ forces $\sum \alpha_j^r \notin S$ if $\sum \alpha_j^r \neq m$. Since $[K : \mathbf{Q}]$ is bounded in terms of m, this divisibility condition on r only depends only on m and S. To handle the possibility $m \in S$, use the positivity hypothesis on the weights to find a prime $\mathfrak{p}_j$ of K over p dividing α_j for each j and let ν be a positive integer large enough so that $p^\nu > m$. Also let e_p and f_p denote the ramification degree and residue field degree at p for the Galois extension $K/\mathbf{Q}$. By taking r to also be divisible by $p^\nu(p^{f_p} - 1)$ we ensure that $\sum \alpha_j^r \bmod \mathfrak{p}_1^{e_p \nu}$ is a sum of m terms each equal to 0 or 1 with at least one such term equal to 0, so by Galois-invariance the rational integer $\sum \alpha_j^r$ is congruent modulo p^ν to a rational integer between 0 and $m - 1$. Thus, due to the choice of ν, we get that $\sum \alpha_j^r \neq m$ for all such divisible r (depending on p and m).

Step 4. Now we turn to the case when $\overline{\mathscr{X}}_c$ has genus 0, which is to say $\deg R_c \leq 2$. If $\deg R_c = 1$ (resp. 2) then $\mathscr{X}_c$ has 1 (resp. 2) points at infinity, so

$$\sum_{f_c(z) \neq 0} \chi_{\kappa''}(R_c(z)) = \begin{cases} -\#\{z \in \kappa'' \mid f_c(z) = 0, R_c(z) \neq 0\}, & \text{if } \deg R_c = 1, \\ -1 - \#\{z \in \kappa'' \mid f_c(z) = 0, R_c(z) \neq 0\}, & \text{if } \deg R_c = 2. \end{cases}$$

Thus, the case $\deg R_c = 2$ is settled, and to handle the (presumably "generic") case $\deg R_c = 1$ we just need that f_c has more than one root in κ''; note that $\deg f_c > 0$ since there is a branch point over c. Since $\deg f_c \leq \deg_T f$, we can assume that f_c is split and so we just need to avoid the case when f_c has one geometric root. Writing $f = \sum \alpha_i T^{pe_i}$ with $(\alpha_i) \in \prod L(\rho_i \cdot \xi)_{\kappa'}$ satisfying $-\mathrm{ord}_\xi(\alpha_i) = \rho_i$, it suffices that for all closed points $c_0 \in C_{\kappa'}$ over which the generically étale projection $Z_f \to \mathbf{A}^1$ has a branch point, the nonzero specialization $f_{c_0} \in \kappa'(c_0)[T]$ has more than one geometric root if it has positive degree.

Working over an algebraic closure k of κ and writing $e_i = p^\mu e_i'$ for all i with a maximal $\mu \geq 0$ (so $p \nmid e_i'$ for some i), it suffices to prove that for a generic k-point $(\alpha_i) \in \prod_i L(\rho_i \cdot \xi)$ in the sense of the Zariski topology, the specialization of the associated polynomial $f = \sum_{i=0}^N \alpha_i T^{e_i'}$ at points of the image in C_k of the branch locus B_f of $Z_f \to \mathbf{A}_k^1$ never has exactly one geometric root. Suppose otherwise, so for *any* nonempty Zariski-open subset $\mathscr{U} \subseteq \prod_i L(\rho_i \cdot \xi)$ there is some $(\alpha_i) \in \mathscr{U}(k)$ and some $c_0 \in C(k)$ (depending on (α_i)) such that

$$\sum \alpha_i(c_0) T^{e_i'} = b(T - t_0)^e$$

with $e > 0$ and $b \in k^\times$. We may and do at least require $\mathscr{U}$ to be small enough so that α_i and α_j have disjoint zero loci on C for $i \neq j$. The only point in the fiber of $Z_f \to C$ over c_0 is the point (c_0, t_0) at which $Z_f \to \mathbf{A}^1$ must therefore have a branch point.

If $t_0 = 0$ then for all but possibly one i we have that α_i vanishes at c_0, so the members of our family must be binomials in T. This forces the family to consist of polynomials of the form $\{\alpha_1 T^{pe_1} + \alpha_0\}$, and we now explain why such cases cannot arise when there is a branch point (c_0, t_0) such that c_0 is a zero of α_0 or α_1. Since α_1 and α_0 have disjoint zero loci, for $f = \alpha_1 T^{pe_1} + \alpha_0$ corresponding to the point $(\alpha_0, \alpha_1) \in \mathscr{U}$ the fiber of $Z_f \to C$ over zeros of α_1 is empty. Hence, we have to consider the possibility of a branch point (c_0, t_0) for $Z_f \to \mathbf{A}^1$ such that $\alpha_0(c_0) = 0$ (so $\alpha_1(c_0) \neq 0$). In such cases we must have $t_0 = 0$, and so the branch condition is that α_0 has a multiple zero at c_0. But for small enough $\mathscr{U}$ the zero scheme of α_0 is étale. Hence, the case $t_0 = 0$ (with sufficiently small $\mathscr{U}$) indeed cannot occur.

Step 5. Now we may suppose $t_0 \neq 0$ (so $\alpha_0(c_0) \neq 0$). We have to separately treat the possibilities that $\alpha_N(c_0)$ is zero or not, so first assume $\alpha_N(c_0) \neq 0$. This forces $e = e'_N$ and $b = \alpha_N(c_0)$, so the resulting identity $\sum \alpha_i(c_0)T^{e'_i} = \alpha_N(c_0)(T - t_0)^{e'_N}$ forces $p|\binom{e'_N}{j}$ whenever $j \notin \{0 = e'_0, \dots, e'_N\}$ and also

$$\binom{e'_N}{e'_i}\alpha_N(c_0) \cdot (-t_0)^{e'_N - e'_i} = \alpha_i(c_0)$$

for all $0 \leq i < N$. In particular, since $e'_0 = 0$ we have $\alpha_0(c_0) = \alpha_N(c_0) \cdot (-t_0)^{e'_N}$, so for $0 < i < N$ we must have

$$\alpha_i^{e'_N}(c_0) = \binom{e'_N}{e'_i}^{e'_N} (\alpha_N^{e'_i}\alpha_0^{e'_N - e'_i})(c_0). \tag{3.8}$$

By taking $\mathscr{U}$ small enough we can ensure that the $N - 1$ rational functions

$$\alpha_i^{e'_N} - \binom{e'_N}{e'_i}^{e'_N} \alpha_N^{e'_i}\alpha_0^{e'_N - e'_i}$$

on C (with $0 < i < N$) have disjoint zero loci away from zeros of α_0 (the exponent $e'_N - e'_i$ might be negative).

The validity of (3.8) for all $0 < i < N$ at some common point $c_0 \in C(k)$ such that $\alpha_0(c_0) \neq 0$ therefore forces $N \leq 2$. If $N = 1$ then $p|\binom{e'_N}{j}$ for all j satisfying $0 < j < e'_N$, and if $N = 2$ then this divisibility holds for all such j except for possibly $j = e'_1$. Since $e'_0 = 0$, $p > 2$, and some e'_i is not divisible by p, we readily get a contradiction except if $N = 1$ and $e'_1 = 1$ or if $N = 2$ and $e'_1 = 1$, $e'_2 = 2$. This second case contradicts the hypothesis that $\deg_T f$ is odd, so our family of polynomials must be of the type $\{\alpha_1 T^{p^r} + \alpha_0\}$ with some $r > 0$. This is the class of families that was specifically ruled out from consideration in the statement of the theorem.

Finally, we treat the case $\alpha_N(c_0) = 0$, so α_N is nonconstant and $\alpha_i(c_0) \neq 0$ for all $i < N$. In particular, $N \geq 2$ because if $N = 1$ then the condition $f_{c_0}(t_0) = 0$ with $\alpha_N(c_0) = 0$ would force $\alpha_0(c_0) = 0$, yet we are taking $\mathscr{U}$ small enough so that α_i and α_j have disjoint zero loci on C for any $i \neq j$. Thus, arguing as above with N replaced by the positive $N - 1$ gives (by taking $\mathscr{U}$ small enough) that $N - 1 \leq 2$, with $(\alpha_1^2 - 4\alpha_0\alpha_2)(c_0) = 0$ if $N - 1 = 2$. If $N = 3$ then we can shrink $\mathscr{U}$ to force α_3 and $\alpha_1^2 - 4\alpha_0\alpha_2$ to have disjoint zero loci on C. Since $\alpha_N(c_0) = 0$, the case $N = 3$ is ruled out and so $N = 2$. Our family of polynomials must therefore be $\{\alpha_2 T^{pe_2} + \alpha_1 T^{p^r} + \alpha_0\}$ for some $r > 0$ such that $p^r < pe_2$. Consider f in this family such that there is a branch point (c_0, t_0) of the map $Z_f \to \mathbf{A}^1$ for which $\alpha_2(c_0) = 0$. We seek a contradiction if f lies in a sufficiently small dense open $\mathscr{U}$ in our family.

At least we may suppose $\alpha_0(c_0) \neq 0$ and $\alpha_1(c_0) \neq 0$ since $\alpha_2(c_0) = 0$, so we solve to get $t_0^{p^r} = -\alpha_0(c_0)/\alpha_1(c_0)$. The branch condition says that for any vector field ∂ on C that is non-vanishing at c_0,

$$(\partial\alpha_2)(c_0)t_0^{pe_2} + (\partial\alpha_1)(c_0)t_0^{p^r} + (\partial\alpha_0)(c_0) = 0.$$

Hence, if we fix a nonzero vector field D on C prior to considering f we get

$$(D\alpha_2)(c_0)t_0^{pe_2} + (D\alpha_1)(c_0)t_0^{p^r} + (D\alpha_0)(c_0) = 0.$$

Raising this to the p^{r-1}th power and using the identity $t_0^{p^r} = -\alpha_0(c_0)/\alpha_1(c_0)$ gives that the rational function

$$(D\alpha_2)^{p^{r-1}}(-\alpha_0/\alpha_1)^{e_2} + ((D\alpha_1)(-\alpha_0/\alpha_1) + D\alpha_0)^{p^{r-1}}$$

vanishes at a zero c_0 of α_2. Since α_2 is nonconstant, this easily gives a contradiction by taking $\mathscr{U}$ small enough. Hence, this case does not occur. ∎

Remark 3.9. Assume $p \neq 2$. It is natural to ask for the analogue of (2.4) for several nonconstant elements $f_1, \ldots, f_r$ in $A[T]$ that are irreducible and pairwise relatively prime in $K[T]$. Assume that $f_1, \ldots, f_s$ lie in $K[T^p]$ and (if $s < r$) $f_{s+1}, \ldots, f_r$ are not in $K[T^p]$. We also assume that $\prod_{j=1}^{s} f_j$ has no local obstructions with respect to A, and we let I_κ be the least common multiple of the polynomials I_{f_j} for $1 \leq j \leq s$. Based on some numerical investigations, we believe the correction factor in degree $n \gg 0$ should be

$$\Lambda_A(f_1, \ldots, f_s; n) := \frac{\sum_{\deg a = n, (f_j(a), I_\kappa)=1} \prod_{j=1}^{s}(|\mu(f_j(a))| - \mu(f_j(a)))}{\sum_{\deg a = n, (f_j(a), I_\kappa)=1} \prod_{j=1}^{s} |\mu(f_j(a))|} \in [0, 2^s] \cap \mathbf{Q}, \tag{3.9}$$

where the condition $\gcd(f_j(a), I_\kappa) = 1$ (a congruence condition on a modulo $\mathrm{Rad}(I_\kappa)$) is imposed for all $1 \leq j \leq s$; we take n large enough so that the denominator in (3.9) is nonzero. (If no f_j's lie in $K[T^p]$ then this correction factor is 1.)

As in the case of a single polynomial, it is not obvious that (3.9) is a periodic function of large n. It is left as an exercise for the interested reader to check that (3.9) satisfies an analogue of Theorem 3.1 (with essentially the same proof): it is periodic in $n \bmod 4$ for large n (depending only on the genus and the total degrees $\deg_{u,T} f_j$ for $j \leq s$), and if we replace the ideal I_κ with any nonzero multiple J in the relative primality conditions on the sums in the numerator and denominator of (3.9) then the resulting function is unaffected in sufficiently large degrees (depending on the genus, the $\deg_{u,T} f_j$'s for $j \leq s$, and $\dim_\kappa(A/J)$).

4. Lifting constructions

In our previous work in genus 0 in [3], the case of characteristic 2 was more subtle than the case of odd characteristic. The key source of difficulties was the need to use lifts to characteristic 0. The necessity of using such lifts for the study of characteristic 2 arises from the following variant on [5, Thm. 3.1], which is essentially due to Swan and was recorded in [3, Thm. 2.4]:

Theorem 4.1. *Let R be a finite étale algebra over a finite field κ of characteristic 2. Let $W = W(\kappa)$ and let $\widetilde{R}$ be the unique lift of R to a finite étale W-algebra. Then*

$$(-1)^{\#\,\mathrm{Spec}(R)} = (-1)^{\dim_\kappa R}\chi(\mathrm{disc}_W \widetilde{R}), \tag{4.1}$$

where $\chi : \kappa^\times \times (1+4W) \twoheadrightarrow \{\pm 1\}$ is the unique quadratic character whose kernel is the index-2 subgroup of elements that are squares in $W^\times$. (Explicitly, χ is given by $c \cdot (1+4w) \mapsto (-1)^{\mathrm{Tr}_{\kappa/\mathbf{F}_2}(w \bmod 2)}$.)

Remark 4.2. To see that (4.1) makes sense, one needs the well-known fact that the 1-unit part of the discriminant of a finite étale W-algebra lies in $1 + 4W$. (The character χ has no natural extension to an order-2 character on $W^\times$.)

In this section we focus our efforts on constructing suitable lifts over p-adic fields, as preparation for proving a 2-adic analogue of [3, Thm. 5.5] in §5 that gives a rigid-analytic factorization of discriminants (rather than an algebraic factorization) over the fraction field of $W(\kappa)$. In §6 we will use this rigid-analytic factorization and Theorem 4.1 "in families" to establish a characteristic-2 analogue of the theory in §3. We also note at the outset that rigid-analytic factorization did not arise in the case of genus 0 with $p = 2$ in [3] because lifting $\mathbf{A}^1_\kappa$ into characteristic 0 can be done very easily and explicitly by using $\mathbf{A}^1_{W(\kappa)}$ (and the generic and closed fibers of $\mathbf{P}^1_{\kappa[u]} \to \operatorname{Spec} \kappa[u]$ have the *same* Weierstrass gap sequence at ∞, namely the empty set).

To keep the role of the finiteness of κ and the parity of $\operatorname{char}(\kappa)$ in perspective, for now we work with an arbitrary perfect field k of positive characteristic p, any smooth and geometrically connected affine k-curve $C = \operatorname{Spec} A$ with one geometric point ξ at infinity, and any primitive polynomial $h \in A[T]$ such that $h(T^p)$ is squarefree in $K[T]$ (so h is also squarefree in $K[T]$), where K is the fraction field of A. Our interest will eventually be in the study of specializations of $h(T^p)$ with $p = 2$. As the reader will see, the parity of $\operatorname{char}(k)$ is irrelevant for the remainder of this section.

We let $F = \operatorname{Frac}(W)$ with $(W, \mathfrak{m}_W)$ a complete mixed-characteristic discrete valuation ring having residue field k. In order to permit the use of certain base-change arguments in later proofs (e.g., the proof of Lemma 6.1 and the end of the proof of Lemma 5.2), it is convenient to not require that W be absolutely unramified (although the absolutely unramified case is the one to which we will apply the theory that we develop below). By [8, III, Cor. 7.4], there exists a proper smooth curve $\overline{\mathscr{C}}$ over $\operatorname{Spec} W$ with closed fiber $\overline{C}$. The generic fiber of $\overline{\mathscr{C}}$ is a geometrically-connected smooth proper curve of genus g over F. The W-smoothness allows us to construct a section $\widetilde{\xi} \in \overline{\mathscr{C}}(W)$ lifting $\xi \in \overline{C}(k)$. Since the divisor $\widetilde{\xi}$ on $\overline{\mathscr{C}}$ is relatively ample over W, $\mathscr{C} = \overline{\mathscr{C}} - \widetilde{\xi}(\operatorname{Spec} W)$ is affine with coordinate ring $\mathscr{A}$ that satisfies $\mathscr{A}/\mathfrak{m}_W\mathscr{A} = A$.

In the special case $g = 0$, if we fix an isomorphism $A \simeq k[u]$ then the rigidity of $\mathbf{P}^1$ ensures that there exists an isomorphism $\overline{\mathscr{C}} \simeq \mathbf{P}^1_W$ that carries $\widetilde{\xi}$ over to ∞ and identifies $\mathscr{A}$ with $W[u]$ lifting the isomorphism $A \simeq k[u]$. This provides a link with the algebraic considerations on the affine line over W in the genus-0 case of characteristic 2 in [3, §5]. In higher genus there are many non-isomorphic choices of $\overline{\mathscr{C}}$, and we will have to choose a lift $(\overline{\mathscr{C}}, \widetilde{\xi})$ very carefully.

The key property we need is that the leading coefficient $a_0 \in A$ of $h \in A[T]$ lifts to $\widetilde{a}_0 \in \mathscr{A}$ with $\operatorname{ord}_{\widetilde{\xi}_F}(\widetilde{a}_0) = \operatorname{ord}_\xi(a_0)$. In general, we can only say $-\operatorname{ord}_{\widetilde{\xi}_F}(\widetilde{a}_0) \geq -\operatorname{ord}_\xi(a_0)$, since the Laurent expansion of $\widetilde{a}_0$ along $\widetilde{\xi}$ in $\widehat{\mathscr{O}}_{\overline{\mathscr{C}},\widetilde{\xi}}[1/\tau] \simeq W[\![\tau]\!][1/\tau]$ (with τ a local generator of the ideal sheaf of the section $\widetilde{\xi}$) may have its initial nonzero coefficients in $\mathfrak{m}_W$ (or, more geometrically, $\widetilde{a}_{0/F}$ may have zeros on $\mathscr{C}_F = \overline{\mathscr{C}}_F - \{\widetilde{\xi}_F\}$ with reduction ξ, and this forces $\widetilde{a}_{0/F}$ to have a higher-order pole at $\widetilde{\xi}_F$ than a_0 has at ξ). Thus, the property we seek for $\widetilde{a}_0$ is that its pole-order along $\widetilde{\xi}$ is constant, or equivalently that $\widetilde{a}_0$ is a generating section of $\mathscr{O}(d_0 \cdot \widetilde{\xi})$ near the support of $\widetilde{\xi}$, where $d_0 = -\operatorname{ord}_\xi(a_0)$. That is, we want $\{1, \widetilde{a}_0\}$ to be a pair of generating sections for the line bundle $\mathscr{O}(d_0 \cdot \widetilde{\xi})$ over all of $\overline{\mathscr{C}}$. The case $d_0 = 0$ is trivial, since then $a_0 \in k^\times$ and we may choose $\widetilde{a}_0 \in W^\times$ to be any lift of a_0. The case $d_0 \geq 2g - 1$ is also trivial (so $g \leq 1$ is settled), since we can apply the theorem on cohomology and base change to $\mathscr{O}(d_0 \cdot \widetilde{\xi})$ to lift the closed-fiber global section a_0 that generates the closed-fiber stalk at ξ. For example, in genus 0 this amounts to the evident fact that a nonzero element in $k[u]$ may be lifted to an element in $W[u]$ with unit leading coefficient. The situation for genus $g > 1$ and $1 \leq d_0 \leq 2g - 2$ will require more work, as we now explain.

In geometric terms, here is a reformulation of the property that we demand for the lifting $(\overline{\mathscr{C}}, \widetilde{\xi})$: we want a proper smooth lifting $\overline{\mathscr{C}}$ of $\overline{C}$ over W such that the finite flat map $a_0 : \overline{C} \to \mathbf{P}^1_k$ of degree $d_0 > 0$ lifts to a finite flat map $\widetilde{a}_0 : \overline{\mathscr{C}} \to \mathbf{P}^1_W$ (necessarily of degree d_0) with respect to which the Cartier-divisor preimage of ∞ in $\overline{\mathscr{C}}$ is of the form $d_0 \cdot \widetilde{\xi}$ for some $\widetilde{\xi} \in \overline{\mathscr{C}}(W)$; such a $\widetilde{\xi}$ necessarily lifts ξ. If $p \le 2g-2$ then it may happen that $p|d_0$, so $a_0 : \overline{C} \to \mathbf{P}^1_k$ may be wildly ramified at ξ (or elsewhere) or it may be inseparable.

It seems probable that a lifting $\widetilde{a}_0$ generally cannot be found for an arbitrary choice of flat deformation $(\overline{\mathscr{C}}, \widetilde{\xi})$ of $(\overline{C}, \xi)$ when $1 \le d_0 \le 2g-2$, for in such cases the natural projection

$$\lambda : \mathrm{Div}^{d_0}_{\overline{\mathscr{C}}/W} \to \mathrm{Pic}^{d_0}_{\overline{\mathscr{C}}/W}$$

is generally not smooth (nor even flat) and so we cannot expect the fiber of λ over the W-point $\mathscr{O}_{\overline{\mathscr{C}}}(d_0 \cdot \widetilde{\xi})$ to admit a W-point lifting an arbitrary choice of k-point (such as a degree-d_0 effective divisor on $\overline{C}$ that is supported on C). Thus, we must expect to have to choose the lift $(\overline{\mathscr{C}}, \widetilde{\xi})$ of $(\overline{C}, \xi)$ at the same time as we choose the lift of the finite map a_0. Here is the solution to our lifting problem for $(\overline{C}, \xi, a_0)$:

Theorem 4.3. *Let X be a proper, smooth, and geometrically connected curve over a perfect field k, and let W be a complete local noetherian ring with residue field k. Fix $\xi \in X(k)$. For any $d_0 > 0$, any finite flat map $f : X \to \mathbf{P}^1_k$ of degree d_0 with $f^{-1}(\infty) = d_0 \cdot \xi$ may be lifted to a finite flat map $\widetilde{f} : \mathscr{X} \to \mathbf{P}^1_W$ with W-smooth $\mathscr{X}$ such that $\widetilde{f}^{-1}(\infty) = d_0 \cdot \widetilde{\xi}$ for some $\widetilde{\xi} \in \mathscr{X}(W)$ lifting ξ.*

Proof. In the equicharacteristic case there is a section to $\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, W$, so pullback along such a section solves the problem. Thus, we may now assume that k has positive characteristic p. We shall use a formal-GAGA argument, shown to us by Q. Liu, that is simpler than our original argument.

Since k is perfect, if f has inseparability-degree p^e with $e \ge 0$ then it follows that f uniquely factors as $f = h \circ \phi_{\mathbf{P}^1_k, e}$ with h separable and $\phi_{\mathbf{P}^1_k, e}$ equal to the e-fold relative Frobenius morphism for the target $\mathbf{P}^1_k$. Since $\phi_{\mathbf{P}^1_k, e}$ is defined by $t \mapsto t^{p^e}$ in terms of a standard coordinate on $\mathbf{P}^1$, by using the map $\mathbf{P}^1_W \to \mathbf{P}^1_W$ defined by $t \mapsto t^{p^e}$ that has fiber $p^e \cdot \infty$ over ∞ we are immediately reduced to studying h instead of f. Thus, we may assume that f is separable.

Let $P = \mathbf{P}^1_k$ and let π_ξ and $\pi_\infty = 1/t$ be local parameters on X and P at ξ and ∞ respectively. Since $f : X \to P$ is a finite map that is totally ramified over ∞, the map on local rings $k[\pi_\infty]_{(\pi_\infty)} = \mathscr{O}_{P,\infty} \to \mathscr{O}_{X,\xi}$ is a finite flat k-algebra map that is described by $\pi_\infty \mapsto u\pi_\xi^{d_0}$ with $u \in \mathscr{O}_{X,\xi}^\times$. Thus, for some monic $g \in k[Z]$ with nonzero constant term we can find an open affine neighborhood $U_\infty = \mathrm{Spec}\, k[\pi_\infty, 1/g(\pi_\infty)]$ around ∞ such that π_ξ and u respectively extend to sections of $\mathscr{O}_X$ and $\mathscr{O}_X^\times$ over $V_\infty = f^{-1}(U_\infty)$ (again denoted π_ξ and u). Since f is separable we may shrink U_∞ so that the finite flat map $V_\infty - \{\xi\} \to U_\infty - \{\infty\}$ induced by f is étale. Let R be the coordinate ring of the affine V_∞, so $u \in R^\times$ and $\mathrm{Spec}(R/\pi_\xi^{d_0} R) = f^{-1}(\infty)$ is equal to $d_0 \cdot \xi$ as Cartier divisors on X. This gives that $R/\pi_\xi R = k$ with support at $\xi \in X(k)$.

Let $\mathscr{X}$ be an arbitrary proper smooth W-scheme that lifts X, and let $\widehat{\mathscr{X}}$ be the formal completion of $\mathscr{X}$ along X. There is a unique open formal subscheme $\widehat{\mathscr{V}}_\infty$ in $\widehat{\mathscr{X}}$ whose special fiber is V_∞, so $\widehat{\mathscr{V}}_\infty = \mathrm{Spf}\, \mathscr{R}$ is a formal affine and $\mathscr{R}/\mathfrak{m}_W \mathscr{R} = R$. Likewise, there is a unique open formal affine $\widehat{\mathscr{U}}_\infty$ in $\widehat{\mathbf{P}}^1_W$ with special fiber U_∞. Let $\widehat{\pi}_\xi, \widehat{u} \in \mathscr{R}$ be lifts of

$\pi_\xi, u \in R$, and let $\widehat{g} \in W[Z]$ be a lift of $g \in k[Z]$ with leading coefficient in $W^\times$, so $\widehat{u}$ lies in $\mathscr{R}^\times$ and the unique continuous W-algebra map

$$W\{\{\widehat{\pi}_\infty\}\} \to \mathscr{R}$$

sending $\widehat{\pi}_\infty$ to $\widehat{u} \cdot \widehat{\pi}_\xi^{d_0}$ carries $\widehat{g}(\widehat{\pi}_\infty)$ to a unit in $\mathscr{R}$ (as may be checked in R). Thus, we get a continuous map of flat adic W-algebras

$$W\{\{\widehat{\pi}_\infty, 1/\widehat{g}(\widehat{\pi}_\infty)\}\} \to \mathscr{R}$$

that has finite flat reduction

$$f^* : k[\pi_\infty, 1/g(\pi_\infty)] = \mathscr{O}(U_\infty) \to \mathscr{O}(V_\infty) = R$$

modulo $\mathfrak{m}_W$ and so is a finite and flat map. In other words, we have constructed a finite flat $\operatorname{Spf} W$-map

$$\widehat{f}_\infty : \widehat{\mathscr{V}}_\infty \to \widehat{\mathscr{U}_\infty}$$

that lifts the finite flat restriction $f_\infty : V_\infty \to U_\infty$ with $\widehat{\mathscr{V}}_\infty$ open in $\widehat{\mathscr{X}}$. Moreover, since $\operatorname{Spec}(R/\pi_\xi R) = \xi \in X(k)$ and π_ξ is not a zero divisor in R, it follows from standard flatness arguments that $\operatorname{Spf}(\mathscr{R}/\widehat{\pi}_\xi\mathscr{R}) = \operatorname{Spf}(W)$ defines a formal W-point $\widehat{\xi}$ of $\widehat{\mathscr{V}}_\infty$ lifting $\xi \in V_\infty(k) \subseteq X(k)$.

By the same argument, if we let $U_0 = P - \{\infty\}$ and $V_0 = X - \{\xi\}$, and let $\widehat{\mathscr{U}_0}$ and $\widehat{\mathscr{V}}_0$ be the unique open formal affines in $\widehat{\mathbf{P}}^1_W$ and $\widehat{\mathscr{X}}$ with special fibers U_0 and V_0 respectively, then we may construct a finite flat $\operatorname{Spf} W$-map

$$\widehat{f}_0 : \widehat{\mathscr{V}}_0 \to \widehat{\mathscr{U}_0}$$

that lifts the finite flat restriction $f_0 : V_0 \to U_0$. The overlap $\widehat{\mathscr{V}} = \widehat{\mathscr{V}}_0 \cap \widehat{\mathscr{V}}_\infty$ with special fiber $V_0 \cap V_\infty$ is thereby realized in two ways as a finite flat covering of the open formal subscheme $\widehat{\mathscr{U}} = \widehat{\mathscr{U}_0} \cap \widehat{\mathscr{U}_\infty} \subseteq \widehat{\mathbf{P}}^1_W$ such that these coverings lift the same map

$$f : V_0 \cap V_\infty \to U_0 \cap U_\infty.$$

However, by the construction of U_∞ this latter map is finite and étale, and so by the uniqueness of infinitesimal deformations of finite étale covers there is a unique automorphism of $\widehat{\mathscr{V}}$ carrying $\widehat{f}_\infty|_{\widehat{\mathscr{V}}}$ to $\widehat{f}_0|_{\widehat{\mathscr{V}}}$. By gluing $\widehat{\mathscr{V}}_0$ to $\widehat{\mathscr{V}}_\infty$ along this isomorphism between the open copies of $\widehat{\mathscr{V}}$ in each space we obtain a new formal smooth W-scheme $\widehat{\mathscr{X}'}$ that is a formal deformation of X (so $\widehat{\mathscr{X}'}$ is W-proper) and admits a flat map $\widehat{f}$ to $\widehat{\mathbf{P}}^1_W$ that lifts f and glues the two flat maps $\widehat{f}_0$ and $\widehat{f}_\infty$. In particular, $\widehat{f}$ is flat. Since $\widehat{f}$ is a map between proper formal W-schemes and it deforms the finite f, $\widehat{f}$ must be finite.

By Grothendieck's formal GAGA [6, 5.1.4], there exists a unique finite $\mathbf{P}^1_W$-scheme $\mathscr{X}'$ that algebraizes the finite formal $\widehat{\mathbf{P}}^1_W$-scheme $\widehat{\mathscr{X}'}$, and $\mathscr{X}'$ must be flat over $\mathbf{P}^1_W$ since it is W-proper and $\widehat{\mathscr{X}'}$ is flat over $\widehat{\mathbf{P}}^1_W$ (via $\widehat{f}$). Thus, $\mathscr{X}'$ is proper and flat over W with special fiber X, and so $\mathscr{X}'$ is a proper smooth W-curve such that there exists a finite flat map $F : \mathscr{X}' \to \mathbf{P}^1_W$ lifting f. We claim that the fiber of F over $\infty \in \mathbf{P}^1(W)$ is $d_0 \cdot \widetilde{\xi}$ for a section $\widetilde{\xi} \in \mathscr{X}'(W)$ that must necessarily lift $\xi \in X(k)$. By formal GAGA it suffices to check this assertion on formal completions along the special fibers over $\operatorname{Spec} W$, and by working locally over $\widehat{\mathbf{P}}^1_W$ the map $\widehat{F}$ is exactly $\widehat{f}_\infty$ over the formal open neighborhood $\widehat{\mathscr{U}_\infty}$ of $\widehat{\infty} \in \widehat{\mathbf{P}}^1_W(\operatorname{Spf} W)$, so $\widehat{F}^{-1}(\widehat{\infty}) = d_0 \cdot \widehat{\xi}$ for the formal W-point $\widehat{\xi}$ of $\widehat{\mathscr{V}}_\infty \subseteq \widehat{\mathscr{X}'}$. ■

Let $(W, \mathfrak{m}_W)$ be a mixed-characteristic complete discrete valuation ring with residue field k, and let F denote its fraction field. Choose a lift $(\overline{\mathscr{C}}, \widetilde{\xi})$ in accordance with Theorem 4.3 for $a_0 = \mathrm{lead}(h) \in A$ with $d_0 = -\operatorname{ord}_\xi(a_0) \geq 0$, so for $\operatorname{Spec} \mathscr{A} = \mathscr{C} := \overline{\mathscr{C}} - \widetilde{\xi}(\operatorname{Spec} W)$ we can pick $\widetilde{a}_0 \in \mathscr{A}$ with exact order $-d_0$ along $\widetilde{\xi}$, which is to say $\{1, \widetilde{a}_0\}$ generates $\mathscr{O}_{\overline{\mathscr{C}}}(d_0 \cdot \widetilde{\xi})$. (If $d_0 = 0$ then we choose $\widetilde{a}_0 \in W^\times$ lifting $a_0 \in k^\times$.) For each lower-degree coefficient c_j of h with a pole of order $\geq 2g$ at ξ, pick a lift of c_j to $\mathscr{A}$ with a pole of constant order along $\widetilde{\xi}$; this is possible since

$$\mathscr{V}_d = \mathrm{H}^0(\overline{\mathscr{C}}, \mathscr{O}_{\overline{\mathscr{C}}}(d \cdot \widetilde{\xi}))$$

is a finite flat W-module whose formation commutes with base change for $d \geq 2g - 1$. For $d \geq 2g - 1$, let

$$\underline{\mathscr{V}}_d = \operatorname{Spec}(\operatorname{Sym}_W \mathscr{V}_d^\vee)$$

be the affine space over $\operatorname{Spec} W$ associated to $\mathscr{V}_d$. Finally, for each nonzero coefficient of h in $V_{2g-1} = \mathscr{V}_{2g-1}/\mathfrak{m}_W \mathscr{V}_{2g-1} = L((2g-1)\xi)$, pick a lift to an element of $\mathscr{V}_{2g-1} \subseteq \mathscr{A}$. Lift vanishing coefficients of h to $0 \in \mathscr{A}$. Using these lifts of the coefficients of h in A to elements of $\mathscr{A}$, let $H \in \mathscr{A}[T]$ be the resulting lift of h. Due to how we picked H, *especially the lead coefficient*, for large d only depending on the total degree $\deg_{u,T} h$ and genus g the evaluation of H carries $\underline{\mathscr{V}}_d^0$ into $\underline{\mathscr{V}}_{\rho(d)}^0$, where

$$\rho(d) := d \cdot \deg_T h + d_0$$

and

$$\underline{\mathscr{V}}_\delta^0 := \underline{\mathscr{V}}_\delta - \underline{\mathscr{V}}_{\delta-1}$$

for all $\delta \geq 2g$.

Definition 4.4. We call the triple $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ an *admissible* lift of $(\overline{C}, \xi, h)$ over W.

The following flatness result will be useful later (and see the proof of [3, Lemma 5.9] for a genus-zero analogue); in the statement we do not require there to be a unique point at infinity because the absence of such a requirement allows us to work locally in the proof.

Theorem 4.5. *Let* $\mathscr{C} = \operatorname{Spec} \mathscr{A}$ *be an arbitrary smooth affine* W*-scheme with geometrically connected nonempty fibers of dimension* 1*. Let* $H \in \mathscr{A}[T]$ *be a polynomial whose reduction* h *over the domain* $A = \mathscr{A}/\mathfrak{m}_W\mathscr{A}$ *satisfies* $\deg h = \deg H > 0$*, and also assume that* h *is primitive in the sense that the specialization* $h_c \in k(c)[T]$ *is nonzero for all* $c \in C = \operatorname{Spec} A$*. Let* K *denote the fraction field of* A*, and assume that* $h(T^p)$ *is squarefree in* $K[T]$*.*

Let $\mathscr{Z} \subseteq \mathscr{C} \times \mathbf{A}^1_W$ *be the zero scheme of* H*. The projection* $\mathscr{Z} \to \mathbf{A}^1_W$ *is quasi-finite and flat, and it is étale away from a closed subset* $\mathscr{B} \subseteq \mathscr{Z}$ *that is quasi-finite over* $\operatorname{Spec} W$*. With its natural scheme structure defined by the Fitting ideal of* $\Omega^1_{\mathscr{Z}/\mathbf{A}^1_W}$*, this branch scheme* $\mathscr{B}$ *is quasi-finite and flat over* W*.*

Proof. The closed fiber of $\mathscr{Z}$ over $\operatorname{Spec} W$ is the zero-scheme $Z = Z_h$ of h. This projects to $\mathbf{A}^1_k$ with finite fibers, since otherwise $h \in K[T]$ would have a root algebraic over k and so h would have an irreducible factor in $k[T]$, contradicting the fact that k is perfect and $h(T^p)$ is assumed to be squarefree. Thus, Z is a reduced scheme of pure dimension 1 with quasi-finite projection to the affine line.

On the generic fiber over $\operatorname{Spec} W$ we claim that $H \in F(\mathscr{C})[T]$ is squarefree without irreducible factors in $F[T]$ (*i.e.*, there are no roots algebraic over F), which is exactly the algebraic translation of the property that $\mathscr{Z}_F$ is reduced and has quasi-finite projection to $\mathbf{A}^1_F$. To prove this, let η be the generic point of the closed fiber C of $\mathscr{C}$, so H has unit leading coefficient as a polynomial over the discrete valuation ring $\mathscr{O}_{\mathscr{C},\eta}$ and hence its irreducible factorization over the function field $F(\mathscr{C}) = \mathrm{Frac}(\mathscr{A}_F)$ may be chosen over $\mathscr{O}_{\mathscr{C},\eta}$ with unit leading coefficients. In particular, any roots of H that are algebraic over F are integral over $F \cap \mathscr{O}_{\mathscr{C},\eta} = W$. It is therefore enough to show that the reduction of H modulo $\mathfrak{m}_\eta$ in $K[T]$ is squarefree and has no roots algebraic over k. The reduction is $h \in K[T]$, which our hypotheses ensure is squarefree and has no roots algebraic over k.

We have now shown that $\mathscr{Z} \to \mathbf{A}^1_W$ is quasi-finite. On fibers over W this map is flat (being a quasi-finite map from a reduced curve to a regular curve over a field), so as long as $\mathscr{Z}$ is W-flat we may conclude from the fiber-by-fiber flatness criterion that $\mathscr{Z}$ is flat over $\mathbf{A}^1_W$. The coordinate ring of $\mathscr{Z}$ is $\mathscr{A}[T]/(H)$. Since $H \in \mathscr{A}$ has mod-$\mathfrak{m}_W$ reduction $h \in A[T]$ that is not a zero divisor, it follows that $\mathscr{A}[T]/(H)$ is torsion-free over W and hence is W-flat.

With $\mathscr{Z} \to \mathbf{A}^1_W$ now shown to be quasi-finite and flat, étaleness of this map at a point is a property that may be checked in fibers over $\mathrm{Spec}\, W$. More specifically, to prove quasi-finiteness (over $\mathrm{Spec}\, W$) of the non-étale locus of $\mathscr{Z} \to \mathbf{A}^1_W$ it is enough to check étaleness over the generic points of $\mathbf{A}^1_k$ and $\mathbf{A}^1_F$ by working on k-fibers and F-fibers respectively. Since $\mathscr{Z}_F$ is reduced and $F(\mathbf{A}^1_W) = F(T)$ has characteristic 0, the situation on F-fibers is clear. For the closed fiber $Z \subseteq C \times \mathbf{A}^1_k$, [5, Thm. 2.6] ensures that the quasi-finite projection $Z \to \mathbf{A}^1_k$ is generically étale.

To verify W-flatness of the quasi-finite branch scheme $\mathscr{B}$ for the map $\mathscr{Z} \to \mathbf{A}^1_W$, we only need to look at the local rings at points of $\mathscr{B}$ in the closed fiber over $\mathrm{Spec}\, W$. We may work locally along the closed fiber of $\mathscr{C}$ to reduce to the case when the invertible sheaf $\Omega^1_{\mathscr{C}/W}$ is globally free, so there is a nowhere vanishing W-linear derivation D on $\mathscr{A}$. We make D act $W[T]$-linearly on $\mathscr{A}[T]$, so the branch scheme $\mathscr{B}$ is cut out by the two conditions $H = DH = 0$ on $\mathscr{C} \times \mathbf{A}^1_W = \mathrm{Spec}\, \mathscr{A}[T]$. Since the local rings of the W-flat $\mathscr{Z} = \mathrm{Spec}\, \mathscr{A}[T]/(H)$ at closed points on its closed fiber over $\mathrm{Spec}\, W$ are all of pure dimension 2 (by the dimension formula for flat maps) and the local rings at closed points of the closed fiber of the subscheme $\mathscr{B}$ of $\mathscr{Z}$ cut out by the element DH have dimension ≤ 1 (as $\mathscr{B}$ is quasi-finite over W) and thus have pure dimension 1 (by Krull's Hauptidealsatz), the element DH must be nowhere a zero divisor along points in the closed fiber of $\mathscr{Z}$. Hence, the Cohen-Macaulay property of $\mathscr{Z}$ (a Cartier divisor on a regular scheme $\mathscr{C} \times \mathbf{A}^1_W$) is inherited by $\mathscr{B}$ along points of its closed fiber over $\mathrm{Spec}\, W$. Since $\mathscr{B} \to \mathrm{Spec}\, W$ is a quasi-finite map from a Cohen-Macaulay scheme to a regular scheme, and the closed-fiber closed points of $\mathscr{B}$ have local rings of the same dimension as W, by [12, 23.1] we conclude that $\mathscr{B}$ is flat over W. ■

Since $\mathscr{B}$ as in Theorem 4.5 is a quasi-finite flat W-scheme and W is a henselian local ring, the structure theorem for quasi-finite separated maps [7, IV_4, 18.5.11] provides a decomposition

$$\mathscr{B} = \mathscr{B}^{\mathrm{f}} \coprod \mathscr{B}' \tag{4.2}$$

where $\mathscr{B}^{\mathrm{f}}$ is finite flat over W (it is Spec of a finite product of finite flat local W-algebras) and $\mathscr{B}'$ has empty closed fiber (*i.e.*, $\mathscr{B}'$ is a finite F-scheme). In particular, on the special fiber over $\mathrm{Spec}\, W$ we have that $\mathscr{B}^{\mathrm{f}} \bmod \mathfrak{m}_W \subseteq C \times \mathbf{A}^1_\kappa$ is the branch scheme B_h for the generically étale projection $Z = Z_h \to \mathbf{A}^1_\kappa$. The importance of the decomposition (4.2) is that it ensures that each point in the closed-fiber branch scheme lifts to a characteristic-0 point of the branch scheme. A typical point of $\mathscr{B}_F$ is denoted $x = (u_x, t_x) \in \mathscr{C}_F \times \mathbf{A}^1_F$ and we functorially define the algebraic map $P_{x,d} : \mathscr{V}_{d/F} \to \mathbf{A}^1_F$ by the norm construction

$$\widetilde{a} \mapsto P_{x,d}(\widetilde{a}) = \mathrm{N}_{R\otimes_F F(x)/R}(\widetilde{a}(u_x) - t_x) \in R \tag{4.3}$$

for any F-algebra R and $\widetilde{a} \in \underline{\mathscr{V}}_d(R)$. Of course, if $x \in \mathscr{B}_F$ lies in the W-finite $\mathscr{B}^{\mathrm{f}}$ then x uniquely extends to a $W(x)$-valued point of $\mathscr{C} \times \mathbf{A}^1_W$ (where $W(x)$ is the valuation ring of $F(x)$), in which case $P_{x,d}$ uniquely extends to a W-morphism $P_{x,d,W} : \underline{\mathscr{V}}_d \to \mathbf{A}^1_W$ defined functorially on W-algebras R by the norm construction analogous to (4.3) using the finite flat extension $W(x)$ over W. Due to W-flatness of $\mathscr{B}$, those $x \in \mathscr{B}_F$ that extend to integral points of the separated W-scheme $\mathscr{C} \times \mathbf{A}^1_W$ are precisely the points in $\mathscr{B}^{\mathrm{f}}_F$.

Norm-functions analogous to the $P_{x,d}$'s provide an algebraic factorization of discriminants in characteristic p in [5, Thm. 4.1]. In §5 we shall use the F-scheme maps in (4.3) to construct rigid-analytic factorizations of discriminants in characteristic 0. The essential new ingredient that was not encountered in the analogous problem in genus 0 in [3, §5] for $p = 2$ is the possibility of unequal Weierstrass gap sequences on the generic and closed fibers of the lifted 2-adic curve $\overline{\mathscr{C}}$, and this is the reason why non-algebraic rigid-analytic factorizations will intervene in our study of periodicity properties of the Möbius function in characteristic 2 for higher genus.

Remark 4.6. For our work over finite fields κ of characteristic 2 it is enough to work with a single (well-chosen) lift $(\overline{\mathscr{C}}, \widetilde{\xi})$ of $(\overline{C}, \xi)$ over $W(\kappa)$, and in what follows it would simplify matters a lot (and is sufficient) to work with such a lift for which the Weierstrass gap sequence at $\widetilde{\xi}$ in characteristic 0 is the same as that at ξ in positive characteristic. We expect that such a lift does not exist in general (we require the lift to be over an absolutely unramified base), but we do not know any example for which such a lift can be proved to not exist. One reason for the difficulty of finding such an example is that every known gap sequence in characteristic $p > 0$ also arises in characteristic 0.

5. Rigid-analytic considerations

With W and F as above, pick an admissible lift $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ of $(\overline{C}, \xi, h)$ over W in the sense of Definition 4.4, and assume that $h \in A[T]$ is primitive and $h(T^p) \in K[T]$ is squarefree as in Theorem 4.5. Let $\mathscr{A}$ be the coordinate ring of the affine open in $\overline{\mathscr{C}}$ complementary to $\widetilde{\xi}$.

Choose a W-basis $\widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_g$ of $\mathscr{V}_{2g-1}$ lifting a basis $\varepsilon_1, \dots, \varepsilon_g$ of $V_{2g-1} = \mathscr{V}_{2g-1}/\mathfrak{m}_W \mathscr{V}_{2g-1}$ such that $\{\varepsilon_1, \dots, \varepsilon_g\}$ is (as in [5, §3]) adapted to the Weierstrass gap sequence at ξ in the sense that $-\mathrm{ord}_\xi(\varepsilon_i)$ is strictly increasing in i. In general we probably cannot pick $(\overline{\mathscr{C}}, \widetilde{\xi})$ and $\{\widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_g\}$ so that $\{\widetilde{\varepsilon}_{i/F}\}$ is similarly adapted to the Weierstrass gaps at $\widetilde{\xi}_F$ on the generic fiber $\overline{\mathscr{C}}_F$, due to the possible failure of cohomology to commute with base change for low-degree line bundles on $\overline{\mathscr{C}}$; see Remark 4.6. We may extend $\{\widetilde{\varepsilon}_i\}_{1 \le i \le g}$ to compatible bases of each $\mathscr{V}_d$ for $d \ge 2g$, since $\mathscr{V}_d$ is a subbundle of codimension 1 in $\mathscr{V}_{d+1}$ over W for $d \ge 2g - 1$ with all $\mathscr{V}_d$'s commuting with base change on W for $d \ge 2g - 1$. We claim that $\{\widetilde{\varepsilon}_i\}_{i \ge 1}$ is a W-module basis of $\mathscr{A}$. Since $\mathscr{V}_d$ is a direct summand of $\mathscr{V}_{d+1}$ over W for $d \ge 2g - 1$, what must be shown is that $\mathscr{A}$ is the rising union of its W-submodules $\mathscr{V}_d$ for large d.

In terms of Weil divisors on the 2-dimensional regular scheme $\overline{\mathscr{C}}$, the nonzero elements of the subring $\mathscr{A}$ inside of the function field $F(\overline{\mathscr{C}})$ are precisely the elements of $F(\overline{\mathscr{C}})^\times$ whose divisor has nonnegative coefficients away from the irreducible component $\widetilde{\xi}$ (corresponding to the unique codimension-1 point lying outside of the open subscheme $\mathrm{Spec}\,\mathscr{A} \subseteq \overline{\mathscr{C}}$).

Likewise, $\mathscr{V}_d - \{0\}$ consists of those elements in $F(\overline{\mathscr{C}})^\times$ whose divisor has nonnegative coefficients away from the $\widetilde{\xi}$-component and whose $\widetilde{\xi}$-component coefficient is $\geq -d$. Taking $d \to \infty$, we arrive at the description $\cup \mathscr{V}_d$ for $\mathscr{A}$ (inside of $F(\overline{\mathscr{C}})$), as desired. We define $\varepsilon_i \in A$ to be the reduction of $\widetilde{\varepsilon}_i$ for all $i \geq 1$, so the $\widetilde{\varepsilon}_i$'s reduce to a system of compatible bases $\{\varepsilon_1, \dots, \varepsilon_{d-g+1}\}$ of the vector spaces $V_d = L(d \cdot \xi)$ inside $A = \mathscr{A}/\mathfrak{m}_W \mathscr{A}$ for $d \geq 2g-1$. Let $V_d^0 = V_d - V_{d-1}$ for $d \geq 2g$, and let $\underline{V}_d$ and $\underline{V}_d^0$ be the associated affine k-varieties for such d.

Let $\{w_1, \dots, w_g\}$ be the Weierstrass gap sequence at ξ on $\overline{C}$ over k. In what follows we consider d large enough (depending only on g and $\deg_{u,T} h$) so that evaluation of H carries $\underline{\mathscr{V}}_d^0$ into $\underline{\mathscr{V}}_{\rho(d)}^0$; here we use the hypotheses on H as a lift of h. For such d and sections $\widetilde{a} \in \underline{\mathscr{V}}_d^0(W)$, we claim that if d is large enough (again, only depending on g and $\deg_{u,T} h$) then $\mathscr{A}/(H(\widetilde{a}))$ is *finite free* over W with basis represented by

$$\{\widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_{\rho(d)+g}\} - \{\widetilde{\varepsilon}_{\rho(d)+w_r+1-g}\}_{1 \leq r \leq g}, \tag{5.1}$$

where $\rho(d)$ is defined above Definition 4.4. Such finite-freeness is useful because if $\widetilde{a} \in \underline{\mathscr{V}}_d^0(W)$ is chosen to lift an arbitrarily chosen $a \in V_d^0 = \underline{V}_d^0(k)$ for such large d then $\mathscr{A}/(H(\widetilde{a}))$ is a finite flat W-algebra lifting the finite k-algebra $A/(h(a))$, exactly as required for applying Theorem 4.1 to compute $\mu(h(a))$ if k is finite of characteristic 2 and $h(a)$ is squarefree in A. It is crucial that $\mathscr{A}/(H(\widetilde{a}))$ has a W-basis represented by the set (5.1) because this set is independent of $\widetilde{a}$. For the purpose of applying Yoneda's lemma to lift the morphism $\underline{V}_d^0 \to \mathbf{A}_k^1$ defined by

$$a \mapsto \mathrm{disc}_{\varepsilon, \rho(d)}(A/h(a))$$

(discriminant defined as a determinant with respect to the basis $\{\varepsilon_1, \dots, \varepsilon_{\rho(d)+g-1}\}$; cf. [5, (3.11)]) to a formal-algebraic morphism $\underline{\mathscr{V}}_d^{0,\wedge} \to \widehat{\mathbf{A}}_W^1$ over W, we need to prove that this finite-freeness result holds much more generally, as follows.

Let W' be any W-algebra whose maximal ideals contain the maximal ideal of W (*e.g.*, a noetherian W-algebra that is separated and complete for the $\mathfrak{m}_W$-adic topology). We pick $\widetilde{a} \in \underline{\mathscr{V}}_d^0(W')$, and we would like to show that $(W' \otimes_W \mathscr{A})/(H(\widetilde{a}))$ is finite and free as a W'-module, with basis represented by (5.1), at least if d is large enough (depending only on g and $\deg_{u,T} h$, not on W'). Since $H(\widetilde{a}) \in \underline{\mathscr{V}}_{\rho(d)}^0(W')$, this is a special case of the following lemma that partially lifts [5, Lemma 3.3] into characteristic 0:

Lemma 5.1. *Let W' be a W-algebra all of whose maximal ideals contain $\mathfrak{m}_W W'$. For $\alpha \in \underline{\mathscr{V}}_\delta^0(W')$ with $\delta \geq 2g$, the W'-algebra $(W' \otimes_W \mathscr{A})/(\alpha)$ is finite and free as a W'-module with basis*

$$\{\widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_{\delta+g}\} - \{\widetilde{\varepsilon}_{\delta+w_r+1-g}\}_{1 \leq r \leq g}. \tag{5.2}$$

This lemma is not generally true if we allow W' to merely be a W-algebra. For example, the case $W' = F$ runs into difficulties if the generic-fiber Weierstrass gap sequence at $\widetilde{\xi}_F$ is different from that on the closed fiber (*i.e.*, the formation of global sections of $\mathscr{O}(d \cdot \widetilde{\xi})$ with $1 \leq d \leq 2g-2$ may not commute with base change). In this sense, Lemma 5.1 is not quite a mixed-characteristic version of [5, Lemma 3.3]; it is, however, the best one can expect in general.

Proof. Since α generates $\mathscr{O}(\delta \cdot \widetilde{\xi}_{W'})$ near $\widetilde{\xi}_{W'}$, and $\mathscr{O}(\delta \cdot \widetilde{\xi})$ restricts to the structure sheaf of $\mathscr{C} = \operatorname{Spec} \mathscr{A}$ over the complement of $\widetilde{\xi}$, we can identify $(W' \otimes_W \mathscr{A})/(\alpha)$ with the global sections of the cokernel of the map $\alpha : \mathscr{O}_{\overline{\mathscr{C}}_{W'}} \to \mathscr{O}_{\overline{\mathscr{C}}_{W'}}(\delta \cdot \widetilde{\xi})$ defined by multiplication by α.

The support of this cokernel is closed in $\overline{\mathscr{C}}_{W'}$ yet it is disjoint from the section $\widetilde{\xi}_{W'}$, so we conclude that $\mathrm{Spec}(W' \otimes_W \mathscr{A})/(\alpha)$ is proper and quasi-finite over W', hence finite over W'.

We will next show that $(W' \otimes_W \mathscr{A})/(a)$ is W'-flat. We may localize at maximal ideals of W' and hence may assume W' is a local W-algebra with local structure map $W \to W'$ and residue field denoted k'. By standard direct limit arguments, we may assume W' is also noetherian and even complete. A finite W'-algebra is therefore isomorphic to its own completion (with respect to the maximal ideal of W'), so we have an isomorphism $(W' \otimes_W \mathscr{A})/(\alpha) \simeq (W' \widehat{\otimes}_W \mathscr{A}^\wedge)/(\alpha)$ where $\mathscr{A}^\wedge$ is the completion of $\mathscr{A}$ with respect to the maximal ideal of W. The completed tensor product $R = W' \widehat{\otimes}_W \mathscr{A}^\wedge$ is noetherian and complete with respect to the topology defined by the maximal ideal of W', so all maximal ideals of R contain the maximal ideal of W'. Moreover, R is visibly flat (and hence faithfully flat) over W'. Consequently, if we use the local flatness criterion for each localization of $W' \widehat{\otimes}_W \mathscr{A}^\wedge$ at a maximal ideal then the W'-flatness of $(W' \widehat{\otimes}_W \mathscr{A}^\wedge)/(\alpha)$ follows from the W'-flatness of $W' \widehat{\otimes}_W \mathscr{A}^\wedge$ and the fact that the reduction $\overline{\alpha}$ in the domain $k' \otimes_k A$ is not 0 (and hence is not a zero divisor in any localization of $R/\mathfrak{m}_{W'} R = k' \otimes_k A$). This shows that $(W' \otimes_W \mathscr{A})/(\alpha)$ is W'-flat.

With $(W' \otimes_W \mathscr{A})/(\alpha)$ now known to be finite and flat over W', as well as obviously of finite presentation (as an algebra, hence as a module due to finiteness), to establish that (5.1) is a W'-basis we may again localize at maximal ideals of W' and it suffices to check the basis condition modulo the maximal ideal of W'. That is, it is enough to treat $(k' \otimes_k A)/(\overline{\alpha})$ for $\overline{\alpha} \in \underline{V}^0_\delta(k')$, where k' is the residue field of the local W-algebra W'. This case follows from [5, Lemma 3.3] (that is stated over the perfect field k but therefore obviously applies over any extension field k'/k). ■

For any W' as in Lemma 5.1 and any $\alpha \in \underline{\mathscr{V}}^0_d(W')$ with $d \geq 2g$, we can use the ordered basis (5.2) to define the discriminant $\mathrm{disc}_{\widetilde{\underline{\varepsilon}}, W'}((W' \otimes_W \mathscr{A})/(\alpha)) \in W'$ as a specific determinant (not merely as an element that is well-defined up to unit-square multiple), and this construction is *functorial* in such W' with *fixed* $\widetilde{\underline{\varepsilon}}$ and d. The cases of interest for us are those noetherian W' that are separated and complete for the topology defined by the maximal ideal of W. For such W' and any *affine* finite-type W-scheme Y, clearly $Y(W') = \varprojlim Y(W'/\mathfrak{m}_W^{n+1} W') = \widehat{Y}(W')$, where $\widehat{Y}$ is the formal scheme over $\mathrm{Spf}(W)$ obtained by completing Y along its closed fiber and $\widehat{Y}(W')$ denotes the set of $\mathrm{Spf}(W')$-points of $\widehat{Y}$ in the category of formal schemes over $\mathrm{Spf}(W)$. Using the special case $Y = \underline{\mathscr{V}}^0_d$, by Yoneda's lemma we see that the above discriminant construction defines a map

$$\mathrm{disc}^\wedge_{\widetilde{\underline{\varepsilon}}, d} : \underline{\mathscr{V}}^{0,\wedge}_d \to \widehat{\mathbf{A}}^1_W = \mathrm{Spf}(W\{\{T\}\}) \tag{5.3}$$

of topologically finite-type formal W-schemes. This map is a formal deformation of the algebraic k-morphism $\mathrm{disc}_{\underline{\varepsilon}, d} : \underline{V}^0_d \to \mathbf{A}^1_k$ defined as in [5, (3.10)]. In particular, if W' is the valuation ring of a finite extension of $F = \mathrm{Frac}(W)$ and we identify $\underline{\mathscr{V}}^0_d(W')$ with $\underline{\mathscr{V}}^{0,\wedge}_d(W')$ then

$$\mathrm{disc}^\wedge_{\widetilde{\underline{\varepsilon}}, d}(\widetilde{a}) = \mathrm{disc}_{\widetilde{\underline{\varepsilon}}, W'}((W' \otimes_W \mathscr{A})/(\widetilde{a}))$$

for $\widetilde{a} \in \underline{\mathscr{V}}^0_d(W')$. It seems unlikely in general (for $g \geq 2$) that there is a map $\underline{\mathscr{V}}^0_d \to \mathbf{A}^1_W$ of ordinary W-schemes that gives rise to $\mathrm{disc}^\wedge_{\widetilde{\underline{\varepsilon}}, d}$ by passage to formal completions, so to lift $\mathrm{disc}_{\underline{\varepsilon}, d} : \underline{V}_d \to \mathbf{A}^1_k$ it seems necessary to use formal schemes as above. (For example, the coordinate ring of $\underline{\mathscr{V}}^0_d$ does not satisfy the requirements on W' in Lemma 5.1.)

Rather than work only with the normal (even regular) formal schemes $\mathrm{Spf}(\mathscr{A}^\wedge) = \mathscr{C}^\wedge$ and $\underline{\mathscr{V}}_d^{0,\wedge}$, it will be necessary to apply Raynaud's "generic-fiber" functor $(\cdot)^{\mathrm{rig}}$ from the category of locally topologically finite-type formal W-schemes to the category of rigid-analytic spaces over F; we refer the reader to [1] for a development of the basic properties of this functor. The formal completion $\mathscr{C}^\wedge$ is open within the proper formal curve $\overline{\mathscr{C}}^\wedge$, and $(\mathscr{C}^\wedge)^{\mathrm{rig}} = \mathrm{Sp}(F \otimes_W \mathscr{A}^\wedge)$ is an admissible open affinoid inside of the proper rigid-analytic curve $(\overline{\mathscr{C}}^\wedge)^{\mathrm{rig}} \simeq \overline{\mathscr{C}}_F^{\mathrm{an}}$ over F (this latter isomorphism is established via the valuative criterion for properness; see [2, 5.3.1(4)]). The choice of $\widetilde{\underline{\varepsilon}}$ allows us to identify the Raynaud generic-fiber $\widetilde{V}_d^0$ of $\underline{\mathscr{V}}_d^{0,\wedge}$ with an affinoid

$$\mathrm{Sp}(F\langle \widetilde{c}_1, \dots, \widetilde{c}_{d+1-g}, 1/\widetilde{c}_{d+1-g}\rangle) \simeq \mathbf{B}^{d-g} \times \partial\mathbf{B},$$

where $\mathbf{B}$ is the closed unit ball over F and $\partial\mathbf{B} = \mathrm{Sp}(F\langle t, 1/t\rangle) = \{|t| = 1\}$ is the "boundary". Let $\widetilde{V}_d = (\underline{\mathscr{V}}_d^\wedge)^{\mathrm{rig}}$, so we have a closed immersion $\widetilde{V}_{d-1} \hookrightarrow \widetilde{V}_d$. Explicitly, $\widetilde{V}_d$ is a unit polydisc on parameters $\widetilde{c}_1, \dots, \widetilde{c}_{d+1-g}$, and $\widetilde{V}_d^0 \subseteq \widetilde{V}_d$ is the locus $|\widetilde{c}_{d+1-g}| = 1$, whereas $\widetilde{V}_d - \widetilde{V}_{d-1}$ is the locus $0 < |\widetilde{c}_{d+1-g}| \le 1$.

We define $\mathrm{disc}^{\mathrm{rig}}_{\widetilde{\underline{\varepsilon}},d} : \widetilde{V}_d^0 \to \mathbf{B}$ to be the Raynaud generic-fiber of $\mathrm{disc}^\wedge_{\widetilde{\underline{\varepsilon}},d}$. For large d (depending only on g and $\deg_{u,T} h$), any finite extension F'/F with valuation ring W', and any $\alpha \in \widetilde{V}_d^0(F') = \underline{\mathscr{V}}_d^{0,\wedge}(W')$, we have that $\mathrm{disc}^{\mathrm{rig}}_{\widetilde{\underline{\varepsilon}},\rho(d)}(H(\alpha)) \in W'$ is a discriminant of the finite F'-algebra

$$\mathscr{A}_{F'}/(H(\alpha)) = F' \otimes_{W'} (\mathscr{A}_{W'}/(H(\alpha)))$$

relative to a W'-basis of the order $\mathscr{A}_{W'}/(H(\alpha))$. Such an "integral" discriminant in the case $p = 2$ is what we will need to use in our study of Möbius-periodicity in characteristic 2.

Pick an F-basis $\underline{\varepsilon}_F = \{\varepsilon_{i,F}\}_{i\ge 1}$ of $\mathscr{A}_F$ adapted to the $\underline{\mathscr{V}}_{d/F}$'s in the sense that we require that $-\mathrm{ord}_{\widetilde{\xi}_F}(\varepsilon_{i,F})$ is strictly increasing in i. Use the choice of $\underline{\varepsilon}_F$ to define the F-scheme morphism $\mathrm{disc}_{\underline{\varepsilon}_F,d} : \underline{\mathscr{V}}^0_{d/F} \to \mathbf{A}^1_F$ as in [5, (3.10)]. It is important to understand how this algebraic discriminant over $\mathrm{Spec}\, F$ is related to the formal discriminant (5.3) over $\mathrm{Spf}(W)$. The relation is given by:

Lemma 5.2. *For $d \ge 2g$, there exists a nonvanishing rigid-analytic function Δ_d on $\widetilde{V}_d^0$ (depending on $\widetilde{\underline{\varepsilon}}$ and $\underline{\varepsilon}_F$) such that*

$$(\mathrm{disc}^\wedge_{\widetilde{\underline{\varepsilon}},d})^{\mathrm{rig}} = \Delta_d^2 \cdot \mathrm{disc}^{\mathrm{an}}_{\underline{\varepsilon}_F,d}|_{\widetilde{V}_d^0}.$$

In the genus-0 case with $\mathscr{A} = W[u]$, if we choose $\varepsilon_{i,F} = u^{i-1}$ and $\widetilde{\varepsilon}_i = u^{i-1}$ then $\Delta_d = 1$. More generally, the proof of Lemma 5.1 can be modified at the end (adapting the appeal to [5, Lemma 3.3]) to avoid the requirement that $-\mathrm{ord}_\xi(\varepsilon_i)$ is strictly increasing in i for $i \le g$ (as long as $\{\varepsilon_1, \dots, \varepsilon_g\}$ is a basis of $L((2g-1)\xi)$), so it is possible to get by with the choice $\varepsilon_{i,F} = \widetilde{\varepsilon}_i$ for all i even though this makes the pole order of the $\varepsilon_{i,F}$'s along $\widetilde{\xi}_F$ fail to be strictly increasing for $i \le g$ if the gap sequences at ξ and $\widetilde{\xi}_F$ are not the same. The crucial point is that even if we make such a convenient-looking choice of $\varepsilon_{i,F}$, the function Δ_d still cannot be expected to be better than rigid-analytic in the case of unequal gap sequences since the determinants on the two sides of the identity in Lemma 5.2 involve different subsets of $\{\widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_{d+g-1}\}$.

Proof. We will construct Δ_d by Yoneda's lemma. Let R be an F-affinoid algebra, so $R = F \otimes_W \mathscr{R}$ for a topologically finite-type and flat adic W-algebra $\mathscr{R}$, and consider a point $\widetilde{a} \in \widetilde{V}_d^0(R)$ having the form $\widetilde{a} = \alpha^{\mathrm{rig}}$ for some (necessarily unique) $\alpha \in \underline{\mathscr{V}}_d^{0,\wedge}(\mathscr{R})$. An important special case is to take α to be the identity point of the functor represented by the affinoid $\widetilde{V}_d^0 = (\underline{\mathscr{V}}_d^{0,\wedge})^{\mathrm{rig}}$. By functoriality of $(\cdot)^{\mathrm{rig}}$ we have

$$(\mathrm{disc}^{\wedge}_{\widetilde{\varepsilon},d})^{\mathrm{rig}}(\widetilde{a}) = \mathrm{disc}^{\wedge}_{\widetilde{\varepsilon},d}(\alpha) = \mathrm{disc}_{\widetilde{\varepsilon},W}((\mathscr{R}\widehat{\otimes}_W\mathscr{A}^{\wedge})/(\alpha)),$$

with this final discriminant computed for the finite flat $\mathscr{R}$-algebra $(\mathscr{R}\widehat{\otimes}_W\mathscr{A}^{\wedge})/(\alpha)$ relative to the basis of $\widetilde{\varepsilon}_i$'s for $i \le g+d$ omitting $\widetilde{\varepsilon}_{d+w_r+1-g}$'s for $1 \le r \le g$. On the other hand, identifying $\underline{\mathscr{V}}^0_{d/F}(\operatorname{Spec} R)$ with $\underline{\mathscr{V}}^{0,\mathrm{an}}_{d/F}(\mathrm{Sp}(R))$, say with $\widetilde{a}^{\mathrm{alg}}$ going over to $\widetilde{a}$, $\mathrm{disc}^{\mathrm{an}}_{\underline{\varepsilon}_F,d}(\widetilde{a})$ is the discriminant of the finite flat R-algebra $(R\otimes_F\mathscr{A}_F)/(\widetilde{a}^{\mathrm{alg}})$ relative to the F-basis of $\varepsilon_{i,F}$'s for $i \le d+g$ omitting $\varepsilon_{d+\widetilde{w}_r+1-g,F}$'s for $1 \le r \le g$ with $\{\widetilde{w}_1,\ldots,\widetilde{w}_g\}$ denoting the Weierstrass gap sequence at $\widetilde{\xi}_F$ on the F-fiber of $\overline{\mathscr{C}}$.

The R-algebra map $R\otimes_F\mathscr{A}_F = F\otimes_W(\mathscr{R}\otimes_W\mathscr{A}) \to F\otimes_W(\mathscr{R}\widehat{\otimes}_W\mathscr{A}^{\wedge})$ carries $\widetilde{a}^{\mathrm{alg}}$ to $1\otimes\alpha$, and passing to the quotient induces a map of R-algebras

$$\theta : (R\otimes_F\mathscr{A}_F)/(\widetilde{a}^{\mathrm{alg}}) \to F\otimes_W((\mathscr{R}\widehat{\otimes}_W\mathscr{A}^{\wedge})/(\alpha)).$$

The source and target have R-bases $\{\varepsilon_{i,F}\}_{i\le d+g, i\ne d+\widetilde{w}_r+1-g}$ and $\{\widetilde{\varepsilon}_i\}_{i\le d+g,i\ne d+w_r+1-g}$ respectively. Thus, as long as θ is an isomorphism in general we may form the determinant of the change-of-basis matrix to get a unit in R. In the universal case with R the coordinate ring of $\widetilde{V}_d^0$, this unit (or its reciprocal) is the desired analytic function Δ_d. The possible failure of the gap sequence $\{w_r\}$ to equal the gap sequence $\{\widetilde{w}_r\}$ (see Remark 4.6) is the reason why Δ_d may not be taken to be identically 1 in general.

Since θ is an R-linear map between finite free R-modules of the same rank, to prove that θ is an isomorphism it is enough to prove surjectivity modulo maximal ideals of R. Every maximal ideal $\mathfrak{m}$ of R has the form $F\otimes_W\mathfrak{p}$ for a prime $\mathfrak{p}$ of $\mathscr{R}$ meeting W in $\{0\}$ with $\dim\mathscr{R}/\mathfrak{p} = 1$ [1, Lemma 3.4], so by functoriality in $\mathscr{R}$ we may replace $\mathscr{R}$ with $\mathscr{R}/\mathfrak{p}$ to reduce to the case in which $R = F'$ is a finite extension of F and $\mathscr{R}$ is W-finite (more specifically, a W-order in F'). We then have an isomorphism

$$F\otimes_W(\mathscr{R}\widehat{\otimes}_W\mathscr{A}^{\wedge}) \simeq F\otimes_W(\mathscr{R}\otimes_W\mathscr{A}^{\wedge}) \simeq F\otimes_W(W'\otimes_W\mathscr{A}^{\wedge})$$

where W' is the valuation ring of F'. Making the base change from W to W' typically increases the absolute ramification degree, but we were careful to not make ramification restrictions on W at the outset in §4 and so we are reduced to the special case $R = F$ and $\mathscr{R} = W$.

Since $\underline{\mathscr{V}}_d^{0,\wedge}(\mathrm{Spf}(W)) = \underline{\mathscr{V}}_d^0(\operatorname{Spec} W)$, α is induced by an algebraic section α^{alg} of $\underline{\mathscr{V}}_d^0$ over $\operatorname{Spec} W$. Clearly α^{alg} has generic fiber $\widetilde{a}^{\mathrm{alg}}$, and $\mathscr{A}^{\wedge}/(\alpha)$ is the $\mathfrak{m}_W$-adic completion of the W-algebra $\mathscr{A}/(\alpha^{\mathrm{alg}})$, so θ is $F\otimes_W(\cdot)$ applied to the natural map from $\mathscr{A}/(\alpha^{\mathrm{alg}})$ to its $\mathfrak{m}_W$-adic completion $\mathscr{A}^{\wedge}/(\alpha)$. This map is an isomorphism since $\mathscr{A}/(\alpha^{\mathrm{alg}})$ is W-finite. ■

Consider $d \ge 2g$ with d sufficiently large (depending only on g and $\deg_{u,T} h$) so that H carries $\underline{\mathscr{V}}_d^0$ into $\underline{\mathscr{V}}^0_{\rho(d)}$. We want to study $\mathrm{disc}^{\mathrm{rig}}_{\widetilde{\varepsilon},\rho(d)}\circ H^{\mathrm{rig}}$ by using the analytification of the algebraic factorization of $\mathrm{disc}_{\widetilde{\varepsilon}_F,\rho(d)}\circ H_F$ over $\operatorname{Spec} F$ given by [5, (4.10)]. First, we need to define several functions. The analytification functor from algebraic F-schemes to rigid spaces over F provides rigid-analytic maps

$$P^{\mathrm{an}}_{x,d} : \underline{\mathscr{V}}^{\mathrm{an}}_{d/F} \to \mathbf{A}^{1,\mathrm{an}}_F \tag{5.4}$$

for any $x \in \mathscr{B}_F$. Since $\underline{\mathscr{V}}^{\mathrm{an}}_{d/F}(\mathrm{Sp}R) = \underline{\mathscr{V}}_d(\operatorname{Spec} R)$ for any F-affinoid algebra R, we have the functorial description $P^{\mathrm{an}}_{x,d}(\widetilde{a}) = \mathrm{N}_{R\otimes_F F(x)/R}(\widetilde{a}(u_x) - t_x)$ for any F-affinoid algebra R and any $\widetilde{a} \in \underline{\mathscr{V}}^{\mathrm{an}}_{d/F}(\mathrm{Sp}R)$. Similarly, for x in the F-fiber $\mathscr{B}^{\mathrm{f}}_F$ of $\mathscr{B}^{\mathrm{f}}$ as in (4.2), by passing to formal completions and Raynaud generic-fibers we see that the W-scheme map $P_{x,d,W} : \underline{\mathscr{V}}_d \to \mathbf{A}^1_W$ induces a rigid-analytic map

$$P^{\mathrm{rig}}_{x,d,W} : \widetilde{V}_d = (\underline{\mathscr{V}}^{\wedge}_d)^{\mathrm{rig}} \to \widehat{\mathbf{A}}^{1,\mathrm{rig}}_W = \mathbf{B} = \mathrm{Sp} F\langle T\rangle. \tag{5.5}$$

It is important that (5.4) and (5.5) are compatible for $x \in \mathscr{B}^{\mathrm{f}}_F$. This compatibility is a special case of the general compatibility of Raynaud's generic-fiber functor with analytification [2, 5.3.1]. To be precise, if Y is any finite-type separated W-scheme with formal completion $\widehat{Y}$ along its closed fiber, there is a functorial quasi-compact open immersion $\widehat{Y}^{\mathrm{rig}} \hookrightarrow Y^{\mathrm{an}}_F$ that is an isomorphism for W-proper Y; applying such functoriality for $Y = \underline{\mathscr{V}}_d$ implies that for $x \in \mathscr{B}^{\mathrm{f}}_F$, restricting $P^{\mathrm{an}}_{x,d}$ to the affinoid subdomain $\widetilde{V}_d$ in $\underline{\mathscr{V}}^{\mathrm{an}}_{d/F}$ gives $P^{\mathrm{rig}}_{x,d,W}$.

On the admissible open $\underline{\mathscr{V}}^{0,\mathrm{an}}_{d/F} \subseteq \underline{\mathscr{V}}^{\mathrm{an}}_{d/F}$ we define $P^{0,\mathrm{an}}_{x,d}$ to be the restriction of $P^{\mathrm{an}}_{x,d}$; this is also the analytification of the restriction $P^0_{x,d}$ of $P_{x,d}$ to $\underline{\mathscr{V}}^0_{d/F}$. When $x \in \mathscr{B}_F$ is an integral point (*i.e.*, lies in the generic fiber of $\mathscr{B}^{\mathrm{f}}$), we can use the algebraic function $P_{x,d,W}$ on $\underline{\mathscr{V}}_d$ over $\mathrm{Spec}\, W$ to define a formal function on $\underline{\mathscr{V}}^{\wedge}_d$ over $\mathrm{Spf}\, W$ that we may restrict to the formal completion $\underline{\mathscr{V}}^{0,\wedge}_d$ of the open $\underline{\mathscr{V}}^0_d$ (or equivalently, we may restrict $P_{x,d,W}$ to $\underline{\mathscr{V}}^0_d$ and then pass to formal completions), and then Raynaud's functor provides us with an analytic function $P^{0,\mathrm{rig}}_{x,d,W}$ on the generic fiber $\widetilde{V}^0_d$ of $\underline{\mathscr{V}}^{0,\wedge}_d$. This latter function is the restriction of $P^{\mathrm{rig}}_{x,d,W}$ to the affinoid subdomain $\widetilde{V}^0_d$ within $\widetilde{V}^0_d$, and so for $x \in \mathscr{B}^{\mathrm{f}}_F$ we have that $P^{0,\mathrm{rig}}_{x,d,W}$ is the restriction of $P^{0,\mathrm{an}}_{x,d}$ to $\widetilde{V}^0_d \subseteq \underline{\mathscr{V}}^{0,\mathrm{an}}_{d/F}$.

Choose $\underline{\varepsilon}_F$ as in Lemma 5.2. By applying [5, Thm. 4.5] to the characteristic-0 triple $(\overline{\mathscr{C}}_F, \widetilde{\xi}_F, H_F)$ and using the algebraic factorization of $\mathrm{disc}_{\widetilde{\underline{\varepsilon}}_F,\rho(d)} \circ H_F$ in [5, (4.10)] in this situation, Lemma 5.2 yields an identity of rigid-analytic meromorphic functions on $\widetilde{V}^0_d$:

$$(\mathrm{disc}_{\widehat{\underline{\varepsilon}},d})^{\mathrm{rig}} \circ H^{\mathrm{rig}} = \Delta_d^2 \cdot b_d \widetilde{c}^{\,e_d}_{d+1-g} \prod_{x \in \mathscr{B}_F} (P^{0,\mathrm{an}}_{x,d}|_{\widetilde{V}^0_d})^{e_x} \cdot \frac{(\mathrm{N}^{\mathrm{an}}_{D,\rho(d)} \circ H^{\mathrm{an}}_F)|_{\widetilde{V}^0_d}}{\mathscr{R}_d(H_F, DH_F)^{\mathrm{an}}|_{\widetilde{V}^0_d}}, \tag{5.6}$$

where $D : \mathscr{A}_F \to \mathscr{A}_F$ is any nonzero F-linear derivation, $e_x \in \mathbf{Z}$ is a suitable positive integer for each $x \in B$, the elements $b_d \in F^{\times}$ and $e_d \in \mathbf{Z}$ may depend on $\underline{\varepsilon}_F$ (and the e_x's, b_d, and e_d do not depend on D), and the functorial norm function $\mathrm{N}_{D,\rho(d)}$ (resp. $\mathscr{R}_d(H_F, DH_F)$) is defined in [5, (4.2)] (resp. [5, (4.4)]). If we choose D to arise from a W-linear self-derivation of $\mathscr{A}$ then the numerator and denominator in the fraction on the right side of (5.6) acquire integral structure (and have restrictions to $\widetilde{V}^0_d$ that arise by the Raynaud construction applied to analogous functorial norms on the formal-scheme side).

The fraction on the right side of (5.6) is *a priori* independent of D, for reasons explained below [5, (4.9)], and we now make the independence of D explicit in an important special case. Consider $a \in V^0_d \subseteq A$ such that the nonzero $h(a^p)$ is squarefree in A. (Such a exists if d is sufficiently large, where largeness only depends on g and $\deg_{u,T} h$.) Since $d \geq 2g$ we can pick $\widetilde{a} \in \mathscr{V}^0_d \subseteq \mathscr{A}$ lifting a, and by Lemma 5.1 the zero-scheme $\mathrm{Spec}\, \mathscr{A}/(H(\widetilde{a}^p))$ is a closed subscheme of $\mathscr{C}$ that is (nonempty and) finite étale over W. Pick $D : \mathscr{A} \to \mathscr{A}$ such that near the W-finite support of $\mathrm{Spec}\, \mathscr{A}/(H(\widetilde{a}^p))$ it is dual to a local generator of the invertible sheaf $\Omega^1_{\mathscr{C}/W}$. Let $\overline{D} : A \to A$ be the (nonzero) reduction of D and define $DH \in \mathscr{A}[T]$ by letting D act on $\mathscr{A}[T]$ as a $W[T]$-linear derivation restricting to D on $\mathscr{A}$. Since $(\overline{D}h)(a^p) = \overline{D}(h(a^p))$ is necessarily a unit in the k-étale $A/(h(a^p))$, $(DH)(\widetilde{a}^p)$ has unit image in $\mathscr{A}/(H(\widetilde{a}^p))$. We conclude that $\mathrm{N}_{(\mathscr{A}/(H(\widetilde{a}^p)))/W}((DH)(\widetilde{a}^p)) \in W^{\times}$, so we may consider the integral ratio

$$\frac{\mathrm{N}_{D,\rho(pd)}(H(\widetilde{a}^p))}{\mathrm{N}_{(\mathscr{A}/(H(\widetilde{a}^p)))/W}((DH)(\widetilde{a}^p))} = \mathrm{N}_{(\mathscr{A}/(H(\widetilde{a}^p)))/W}\left(\frac{D(H(\widetilde{a}^p))}{(DH)(\widetilde{a}^p)}\right) \in W$$

that is visibly independent of D. This obviously has reduction 1 in k. Note that this entire setup is compatible with flat local base change $W \to W'$ to another complete discrete valuation ring.

Theorem 5.3. *Pick d large as above, and consider $a \in \underline{V}_d^0(k)$ such that the nonzero $h(a^p)$ is squarefree in A. Define the rational 1-form*

$$\omega_{h,a} = \frac{(\partial_T h)(a^p)a^{p-1}}{h(a^p)}\,\mathrm{d}a$$

on $\overline{C}$. Choose a W-linear derivation $D : \mathscr{A} \to \mathscr{A}$ lifting a k-linear derivation $\overline{D} : A \to A$ that is dual to a local generator of $\Omega^1_{C/k}$ near zeros of $h(a^p)$ on C.

The 1-form $\omega_{h,a}$ has at worst simple poles on $\overline{C}$, and if $p = 2$ then for sufficiently large d

$$\mathrm{N}_{(\mathscr{A}/(H(\widetilde{a}^p)))/W}\left(\frac{D(H(\widetilde{a}^p))}{(DH)(\widetilde{a}^p)}\right) \equiv 1 - p\,\mathrm{ord}_\xi(a)\deg_T(h) + p^2 \sum_{\{y_1,y_2\}} \mathrm{Res}_{y_1}\omega_{h,a}\mathrm{Res}_{y_2}\omega_{h,a} \bmod p^2\mathfrak{m}_W, \tag{5.7}$$

where $\{y_1, y_2\}$ runs over all unordered pairs of distinct geometric poles of $\omega_{h,a}$ on $\overline{C}$ and $\widetilde{a} \in \underline{\mathscr{V}}_d^0(W)$ lifts a. The largeness of d is determined by g and $\deg_{u,T} h$.

Due to the residue theorem and the fact that the residue characteristic is 2 in (5.7), we can include pairs $y_1 = y_2$ in the residue sum in (5.7) without affecting the value of the sum. The congruence (5.7) was proved in [3, Thm. 5.5] in the genus-0 case, with $\mathscr{A} = W(k)[u]$ and $D = \partial_u$. The reader may easily check that our proof of (5.7) works modulo p^2 for any prime p, but (5.7) is only useful for us because for $p = 2$ it holds modulo $p^2\mathfrak{m}_W$.

Proof. Since $h(a^p)$ is squarefree in A, it is obvious that $\omega_{h,a}$ has at worst simple poles away from ξ. The pole-order of $(\partial_T h)(a^p)a^p$ at ξ cannot exceed that of $h(a^p)$ when $d = -\mathrm{ord}_\xi(a)$ is sufficiently large (as determined by $\deg_{u,T} h$), so it is clear that $(\partial_T h)(a^p)a^p/h(a^p)$ cannot have a pole at ξ for such large d, and hence $\mathrm{ord}_\xi(\omega_{h,a}) \geq -1$ for such large d. In what follows, we shall take $-\mathrm{ord}_\xi(a)$ large as just required (in particular, $a \neq 0$). We also take d so large (depending only on g and $\deg_{u,T} h$) that evaluation of H carries $\underline{\mathscr{V}}_d^0$ into $\underline{\mathscr{V}}_{\rho(d)}^0$.

To establish (5.7), we will not need that $p = 2$ until near the end of the proof. Thus, we initially work with a general prime p. Making a base change to the completion of a strict henselization of W, we may reduce to the case of algebraically closed k. Let $\overline{c} \in C(k)$ run over the zeros of the nonzero non-unit $h(a^p) \in A$, and let $c \in \mathscr{C}(W)$ run over the zeros of $H(\widetilde{a}^p)$, so each c lifts a unique $\overline{c}$ since $\mathscr{A}/(H(\widetilde{a}^p))$ is finite étale over W. Clearly $(DH)(\widetilde{a}^p) \in \mathscr{A}$ has unit value at each c, since its reduction $(\overline{D}h)(a^p) = \overline{D}(h(a^p))$ on C has nonzero value at each zero $\overline{c}$ of $h(a^p)$ (because $h(a^p)$ is squarefree and $\overline{D}$ is dual to a local generator of $\Omega^1_{C/k}$ near each $\overline{c}$). By using the Chain Rule to compute $D(H(\widetilde{a}^p))$ we have

$$\frac{D(H(\widetilde{a}^p))}{(DH)(\widetilde{a}^p)}\Big|_c = 1 + p \cdot \frac{(\partial_T H)(\widetilde{a}^p) \cdot \widetilde{a}^{p-1} \cdot (D\widetilde{a})}{(DH)(\widetilde{a}^p)}\Big|_c. \tag{5.8}$$

The product of these 1-units over all c is the left side of (5.7).

Let $P = H(\widetilde{a}^p)$. For each $c \in \mathscr{C}(W)$ as above, P is a local parameter at c_F on the generic fiber $\overline{\mathscr{C}}_F$ since $\mathscr{A}_F/(H(\widetilde{a}^p))$ is finite étale over F. Thus,

$$p \cdot \frac{(\partial_T H)(\widetilde{a}^p)\cdot \widetilde{a}^{p-1}\cdot (D\widetilde{a})}{(DH)(\widetilde{a}^p)}\Big|_c = p \cdot \mathrm{Res}_c \frac{(\partial_T H)(\widetilde{a}^p)\widetilde{a}^{p-1}\cdot D\widetilde{a}}{(DH)(\widetilde{a}^p)}\cdot \frac{\mathrm{d}P}{P}. \tag{5.9}$$

Since $\mathrm{d}P = (dH)(\widetilde{a}^p) + p(\partial_T H)(\widetilde{a}^p)\widetilde{a}^{p-1}\mathrm{d}\widetilde{a}$, where $\mathrm{d}H$ denotes the application of d to the coefficients of $H \in \mathscr{A}[T]$ (just like our convention for defining DH by extending D to a $W[T]$-linear derivation of $\mathscr{A}[T]$), expanding out $\mathrm{d}P/P$ in (5.9) yields

$$p\cdot \mathrm{Res}_c \frac{(\partial_T H)(\widetilde{a}^p)\widetilde{a}^{p-1}\cdot D\widetilde{a}}{H(\widetilde{a}^p)}\cdot \frac{(\mathrm{d}H)(\widetilde{a}^p)}{(DH)(\widetilde{a}^p)} + p^2\cdot \mathrm{Res}_c \left(\frac{(\partial_T H)(\widetilde{a}^p)\widetilde{a}^{p-1}(D\widetilde{a})}{(DH)(\widetilde{a}^p)}\right)^2\cdot \frac{(DH)(\widetilde{a}^p)\cdot \mathrm{d}\widetilde{a}}{H(\widetilde{a}^p)\cdot D\widetilde{a}}$$

(note that $D\widetilde{a} \neq 0$ since $\widetilde{a} \notin F$ and F has characteristic 0). Hence, the right side of (5.8) is

$$1 + p\cdot \mathrm{Res}_c \frac{(\partial_T H)(\widetilde{a}^p)\widetilde{a}^{p-1}\cdot \mathrm{d}\widetilde{a}}{H(\widetilde{a}^p)} + p^2\cdot \mathrm{Res}_c \left(\frac{(\partial_T H)(\widetilde{a}^p)\widetilde{a}^{p-1}(D\widetilde{a})}{(DH)(\widetilde{a}^p)}\right)^2\cdot \frac{(\mathrm{d}H)(\widetilde{a}^p)}{H(\widetilde{a}^p)}, \tag{5.10}$$

due to the identity

$$\frac{(\mathrm{d}H)(\widetilde{a}^p)}{\mathrm{d}\widetilde{a}} = \frac{(DH)(\widetilde{a}^p)}{D\widetilde{a}}$$

of meromorphic 1-forms on $\overline{\mathscr{C}}$; this identity follows from the more precise identity $\mathrm{d}\alpha/\mathrm{d}\widetilde{a} = D\alpha/D\widetilde{a}$ for each coefficient α of $H \in \mathscr{A}[T]$. This latter identity is an immediate consequence of the universal property of $\mathrm{d} : \mathscr{A} \to \Omega^1_{\mathscr{A}/W}$.

Consider the product of (5.10) over the finitely many $c \in \mathscr{C}(W)$ as considered above. We are interested in computing this product modulo $p^2\mathfrak{m}_W$, so it suffices to compute the final residue term in (5.10) modulo $\mathfrak{m}_W$, or in other words as a residue in k at the reduction $\overline{c} \in C(k)$. In characteristic p we have $(\mathrm{d}h)(a^p) = \mathrm{d}(h(a^p))$, and $h(a^p)$ is a local coordinate at $\overline{c}$. Hence, (5.10) modulo $p^2\mathfrak{m}_W$ is

$$1 + p\cdot \mathrm{Res}_c \frac{(\partial_T H)(\widetilde{a}^p)\widetilde{a}^p}{H(\widetilde{a}^p)}\cdot \frac{\mathrm{d}\widetilde{a}}{\widetilde{a}} + p^2\cdot \mathrm{Res}_{\overline{c}} \left(\frac{(\partial_T h)(a^p)a^{p-1}(\overline{D}a)}{(\overline{D}h)(a^p)}\right)^2\cdot \frac{\mathrm{d}(h(a^p))}{h(a^p)} \bmod p^2\mathfrak{m}_W.$$

Note that the first residue term lies in W and the second lies in k. Define

$$\omega_{H,\widetilde{a}} := \frac{(\partial_T H)(\widetilde{a}^p)\widetilde{a}^p}{H(\widetilde{a}^p)}\cdot \frac{\mathrm{d}\widetilde{a}}{\widetilde{a}},\ \ \eta_{h,a} := \left(\frac{(\partial_T h)(a^p)a^{p-1}(\overline{D}a)}{(\overline{D}h)(a^p)}\right)^2\cdot \frac{\mathrm{d}(h(a^p))}{h(a^p)}.$$

Since a and $h(a^p)$ are nonzero in A, the rational 1-form $\omega_{H,\widetilde{a}}$ on $\overline{\mathscr{C}}$ is regular near the generic point of the closed fiber. Thus, the reduction of $\omega_{H,\widetilde{a}}$ to a meromorphic 1-form on C makes sense and is given by using h and a in the respective roles of H and $\widetilde{a}$. More importantly, this reduction process is compatible with reduction of residues along the generic and closed fibers of any section $c \in \mathscr{C}(W)$ that is a zero of $H(\widetilde{a}^p)$ since $\mathscr{A}/(H(\widetilde{a}^p))$ is finite étale over the strictly henselian W. That is, $\mathrm{Res}_c(\omega_{H,\widetilde{a}}) \in W$ reduces to $\mathrm{Res}_{\overline{c}}(\omega_{h,a}) \in k$ for such c. We conclude that the norm in (5.7) is congruent to

$$1 + p\sum_c \mathrm{Res}_c\, \omega_{H,\widetilde{a}} + p^2 \sum_{c_1\neq c_2} \mathrm{Res}_{c_1}\, \omega_{H,\widetilde{a}}\cdot \mathrm{Res}_{c_2}\, \omega_{H,\widetilde{a}} + p^2 \sum_{\overline{c}} \mathrm{Res}_{\overline{c}}\, \eta_{h,a} \bmod p^2\mathfrak{m}_W, \tag{5.11}$$

where the generic-fiber geometric points $c_F \in \overline{\mathscr{C}}(\overline{F})$ in the first two sums in (5.11) run over all points in the geometric generic fiber of the support of the W-finite étale $\mathrm{Spec}\,\mathscr{A}/(H(\widetilde{a}^p))$, which is to say (by looking at the definition of $\omega_{H,\widetilde{a}}$) that they run over all geometric poles of $\omega_{H,\widetilde{a}}$ on $\overline{\mathscr{C}}_F$ except for the unique point $\widetilde{\xi}_F$ complementary to $\mathscr{C}_F = \mathrm{Spec}\,\mathscr{A}_F$ if this point is a pole. Also, the final sum in (5.11) runs over the (pairwise distinct) reductions of these points. By the residue theorem over F, we conclude that the first sum over c's in (5.11) equals $-\mathrm{Res}_{\widetilde{\xi}_F}\omega_{H,\widetilde{a}}$. Since F has *characteristic* 0 and $(\overline{\mathscr{C}},\widetilde{\xi},H)$ satisfies Definition 4.4, by taking d sufficiently large (with largeness only depending on $\deg_{u,T} h$) we see that the residue of $\omega_{H,\widetilde{a}}$ at $\widetilde{\xi}_F$ is equal to $\mathrm{ord}_{\widetilde{\xi}_F}(\widetilde{a})\cdot\deg_T H$, and since $\widetilde{a}$ is a W-section of $\underline{\mathscr{V}}_d^0$ we can write this residue-value as $\mathrm{ord}_\xi(a)\deg_T h$.

Since $(\overline{D}h)(a^p)=\overline{D}(h(a^p))$ in characteristic p, we have

$$\eta_{h,a}=\left((\partial_T h)(a^p)a^{p-1}\frac{\mathrm{d}a}{\mathrm{d}(h(a^p))}\right)^2\cdot\frac{\mathrm{d}(h(a^p))}{h(a^p)}.$$

If we use the residue theorem to express the final sum $\sum_{\overline{c}}\mathrm{Res}_{\overline{c}}\,\eta_{h,a}$ in (5.11) as the negative of the sum of residues of $\eta_{h,a}$ at its poles on $\overline{C}$ away from the $\overline{c}$'s, there is no reason in general to expect the residue at ξ to be the only contribution from poles away from the $\overline{c}$'s; $\eta_{h,a}$ probably has many poles on C away from the $\overline{c}$'s. However, when $p=2$ a miracle happens: the residue at any such pole must be 0! To see this, recall that for a rational point x and a meromorphic 1-form $s^p\mathrm{d}r/r$ on a smooth algebraic curve in characteristic $p>0$, $\mathrm{Res}_x(s^p\mathrm{d}r/r)=(\mathrm{Res}_x(s\mathrm{d}r/r))^p$. Thus, in characteristic $p=2$, for any $x\in C(k)$ where $h(a^p)$ is *nonvanishing*, the residue $\mathrm{Res}_x\,\eta_{h,a}$ is the square of

$$\mathrm{Res}_x(\partial_T h)(a^p)a^{p-1}\frac{\mathrm{d}a}{\mathrm{d}(h(a^p))}\cdot\frac{\mathrm{d}(h(a^p))}{h(a^p)}=\mathrm{Res}_x(\partial_T h)(a^p)a^{p-1}\frac{\mathrm{d}a}{h(a^p)}=0.$$

It follows that when $p=2$,

$$\begin{aligned}\sum_{\overline{c}}\mathrm{Res}_{\overline{c}}\,\eta_{h,a}=-\mathrm{Res}_\xi\,\eta_{h,a}\ &=\ -\left(\mathrm{Res}_\xi\frac{(\partial_T h)(a^p)a^p}{h(a^p)}\cdot\frac{\mathrm{d}a}{a}\right)^2\\ &=\ -(\mathrm{Res}_\xi\omega_{h,a})^2\\ &=\ \mathrm{Res}_\xi\omega_{h,a}\cdot\sum_{\overline{c}}\mathrm{Res}_{\overline{c}}\,\omega_{h,a},\end{aligned}$$

the final equality following from the residue theorem for $\omega_{h,a}$ on $\overline{C}$. Putting everything together, taking the product of the left side of (5.8) over all c's yields (5.7) when $p=2$. ■

Motivated by (5.7) and following [3], we shall use the following notation:

Definition 5.4. For $a\in A$, let $\omega_{h,a}=((\partial_T h)(a^p)a^{p-1}/h(a^p))\,\mathrm{d}a$. For a meromorphic 1-form ω on $\overline{C}$, let $s_2(\omega)$ be the second symmetric function of the residues of ω, indexed by the geometric poles.

Note that Definition 5.4 makes sense without requiring $h(a^p)$ to be squarefree. Clearly $s_2(\omega_{h,a^p})=0$ since $\omega_{h,a^p}=0$. Also, if h is a polynomial in T^p (so $\partial_T h=0$ and hence $\omega_{h,a}=0$ for all a) then we have $s_2(\omega_{h,a})=0$ for all a. As we noted in [3, §5], $s_2(\omega)$ is not "algebraic" in ω if we do not restrict ω to have an étale polar divisor with a fixed degree. The cases of most interest to us will be 1-forms with such a divisor.

In the case $p=2$ with large d as required for (5.7), we shall combine the congruence (5.7) and the rigid-analytic factorization obtained by evaluating (5.6) on $\widetilde{V}_{pd}^0$ at $\widetilde{a}^p$ for points $\widetilde{a}$ of $\widetilde{V}_d^0$. More generally, allowing any p and taking d large as required for (5.6), consider the contribution to (5.6) by $P_{x,d}^{0,\mathrm{an}}$ for $x\in\mathscr{B}_F$. If $x\in\mathscr{B}_F^{\mathrm{f}}$ then we have a compatibility of (5.4) and (5.5): the restriction of $P_{x,d}^{0,\mathrm{an}}$ to an analytic function on $\widetilde{V}_d^0$ is equal to the Raynaud generic-fiber $P^{0,\mathrm{rig}}_{x,d,W}$ of the map $P^{0,\wedge}_{x,d,W} : \underline{\mathscr{V}}^{0,\wedge}_d \to \widehat{\mathbf{A}}^1_W$. If $x = (u_x, t_x) \notin \mathscr{B}^{\mathrm{f}}_F$ then either $|t_x| > 1$ or $u_x \in \mathscr{C}(\overline{F}) \subseteq \overline{\mathscr{C}}(\overline{F}) = \overline{\mathscr{C}}(\overline{W})$ is not a $\overline{W}$-point of $\mathscr{C}$ (where $\overline{W}$ is the valuation ring of an algebraic closure $\overline{F}$ of F), which is to say that either $|t_x| > 1$ or u_x has reduction $\xi \in \overline{C}(k)$. We shall describe this second possibility by writing "$|u_x| > 1$" (since in the genus-0 case $\mathscr{A} = W[u]$, this inequality on the absolute value of the standard coordinate of u_x in $\mathscr{C}_F = \mathbf{A}^1_F$ is exactly the condition that u_x has reduction $\xi = \infty \in \mathbf{P}^1(k)$). We will likewise write "$|u_x| \leq 1$" when u_x does extend to an integral point of $\mathscr{C}$.

The factor $P^{0,\mathrm{an}}_{x,d}|_{\widetilde{V}^0_d}$ in (5.6) for points $x \in \mathscr{B}_F$ with $|u_x| > 1$ can be absorbed into Δ_d when d is a large multiple of p and we evaluate on pth powers $\widetilde{a}^p$, due to:

Lemma 5.5. *If $x \in \mathscr{B}_F$ with $|u_x| > 1$ then the restriction of $P^{0,\mathrm{an}}_{x,d}$ to $\widetilde{V}^0_d$ is nonvanishing on $\widetilde{V}^0_d$ when d is sufficiently large. Moreover, for sufficiently large d, the analytic function $\widetilde{a} \mapsto P^{0,\mathrm{an}}_{x,pd}(\widetilde{a}^p)$ on $\widetilde{V}^0_d$ is the pth power of a nonvanishing analytic function on $\widetilde{V}^0_d$. If $W = W(k)$ then this largeness for d only depends on g and $\deg_{u,T} h$.*

See [3, Thm. 5.7] for a simpler genus-0 analogue with $p = 2$.

Proof. For F-affinoid algebras F' and $\widetilde{a} \in \widetilde{V}^0_d(F')$, we have the factorization

$$\begin{aligned} P^{0,\mathrm{an}}_{x,d}(\widetilde{a}) &= \mathrm{N}_{F'\otimes_F F(x)/F'}(\widetilde{a}(u_x) - t_x) \\ &= \mathrm{N}_{F'\otimes_F F(x)/F'}(\widetilde{a}(u_x)) \cdot \mathrm{N}_{F'\otimes_F F(x)/F'}(1 - t_x\widetilde{a}(u_x)^{-1}) \in F' \end{aligned}$$

if $\widetilde{a}(u_x) \in F' \otimes_F F(x)$ is a unit. We will show that for any $N > 0$, if we make d large enough then for any finite extension F'/F and $\widetilde{a} \in \widetilde{V}^0_d(F')$, the image of $\widetilde{a}(u_x)$ in each factor field of $F' \otimes_F F(x)$ is not only nonzero, but has absolute value $> N$ (all absolute values on finite extensions of F are required to extend the standard absolute value on $\mathbf{Q}_p$); the dependencies for such largeness of d will be addressed later. Grant this for now, so for such large d the analytic function $P^{0,\mathrm{an}}_{x,d}$ is nonvanishing on $\widetilde{V}^0_d$ and, for $\widetilde{a} \in \widetilde{V}^0_d(F')$ with F-affinoid F',

$$P^{0,\mathrm{an}}_{x,pd}(\widetilde{a}^p) = \mathrm{N}_{F'\otimes_F F(x)/F'}(\widetilde{a}(u_x))^p \mathrm{N}_{F'\otimes_F F(x)/F'}(1 - \widetilde{a}(u_x)^{-p} t_x).$$

Define $\mathbf{q} = p^2$ if p is odd and $\mathbf{q} = 8$ if $p = 2$. The p-adic logarithm $t \mapsto \log_p(1 + \mathbf{q}t)$ and p-adic exponential $t \mapsto \exp_p(\mathbf{q}t)$ on the closed unit disc over $\mathbf{Q}_p$ provide canonical pth roots of analytic functions of the form $1 + \mathbf{q}f$ in affinoid algebras over any analytic extension field of $\mathbf{Q}_p$ when $\|f\|_{\sup} \leq 1$. Thus, it would follow that $P^{0,\mathrm{an}}_{x,pd}((\cdot)^p)$ is the pth power of a nonvanishing analytic function on $\widetilde{V}^0_d$ as long as we use d adapted to a choice of N satisfying $N^p \geq |t_x/\mathbf{q}|$, since then $\widetilde{a} \mapsto 1 - \widetilde{a}(u_x)^{-p} t_x$ has a canonical pth root as an analytic function $(\widetilde{V}^0_d)_{F(x)} \to \mathbf{B}^1_{F(x)}$.

We may make a (typically ramified) finite extension of F so that all x's are F-rational, and our problem is to give a large universal lower bound on $|\widetilde{a}(u_x)|$ for $\widetilde{a} \in \widetilde{V}^0_d(F')$ with F' a varying finite extension of F and d fixed but large. Let W' denote the valuation ring of F' and note that $\widetilde{V}^0_d(F') = \underline{\mathscr{V}}^{0,\wedge}_d(\operatorname{Spf} W') = \underline{\mathscr{V}}^0_d(\operatorname{Spec} W')$. The point $u_x \in \mathscr{C}^{\mathrm{an}}_F \subseteq \overline{\mathscr{C}}^{\mathrm{an}}_F = \overline{\mathscr{C}}^{\wedge,\mathrm{rig}}$ has reduction $\xi \in \overline{C}(k)$. Using Berthelot's generalization of Raynaud's "rigid generic fiber" construction (see [9, §7], especially [9, 7.2.5]), the preimage of ξ under the specialization mapping $\mathrm{sp} : \overline{\mathscr{C}}^{\mathrm{an}}_F \simeq \overline{\mathscr{C}}^{\wedge,\mathrm{rig}} \to \overline{C}$ is an admissible open in $\overline{\mathscr{C}}^{\wedge,\mathrm{rig}}$ that is compatibly identified with the Berthelot generic fiber of the formal completion of $\overline{\mathscr{C}}^{\wedge}$ along ξ, which is to say

$\mathrm{sp}^{-1}(\xi) = \mathrm{Spf}(\mathscr{O}^{\wedge}_{\overline{\mathscr{C}}^{\wedge},\xi})^{\mathrm{rig}}$ as admissible opens in $\overline{\mathscr{C}}^{\mathrm{an}}_F \simeq \overline{\mathscr{C}}^{\wedge,\mathrm{rig}}$. This admissible open $\mathrm{sp}^{-1}(\xi)$ is an open unit disc since $\overline{C}$ is smooth at ξ.

Explicitly, the section $\widetilde{\xi} \in \overline{\mathscr{C}}(W) = \overline{\mathscr{C}}^{\wedge}(W)$ lifts ξ, so if we choose a local generator $y \in \mathscr{O}_{\overline{\mathscr{C}},\xi}$ for the ideal of the section $\widetilde{\xi}$ then since W is complete we get a topological isomorphism

$$\mathscr{O}^{\wedge}_{\overline{\mathscr{C}}^{\wedge},\xi} \simeq W[\![y]\!]$$

where $W[\![y]\!]$ has its maximal-adic topology. This identifies $\mathrm{sp}^{-1}(\xi) \subseteq \overline{\mathscr{C}}^{\mathrm{an}}_F$ with an open unit disc $\{|y| < 1\}$, so the point u_x with reduction ξ corresponds to some value $y(u_x) \in \mathfrak{m}_W$. Since the Berthelot and Raynaud constructions are compatible with base change on W, the meromorphic function defined by $\widetilde{a} \in \widetilde{V}^0_d(F') = \underline{\mathscr{V}}^{0,\wedge}_d(W')$ on this open disc has a formal Laurent series

$$\widetilde{a} = c_{d+1-g}(\widetilde{a})y^{-d} + c_{d+g}(\widetilde{a})y^{-d+1} + \cdots \in \mathrm{Frac}(\mathscr{O}^{\wedge}_{\overline{\mathscr{C}}^{\mathrm{an}}_{F'},\widetilde{\xi}_{F'}}) \simeq \mathrm{Frac}(\mathscr{O}^{\wedge}_{\overline{\mathscr{C}}_{F'},\widetilde{\xi}_{F'}}) \simeq F'(\!(y)\!)$$

with a $W'^{\times}$-coefficient in degree $-d$ and W'-coefficients in all other degrees. That is, this power series lies in $W'[\![y]\!][1/y]^{\times}$. Thus,

$$\widetilde{a}(u_x) = y(u_x)^{-d}(c_{d+1-g}(\widetilde{a}) + c_{d+g}(\widetilde{a})y(u_x) + \dots)$$

with $|y(u_x)| < 1$, so $|\widetilde{a}(u_x)| = |y(u_x)|^{-d}$. Since $|y(u_x)|^{-1} > 1$ and $y(u_x)$ has nothing to do with $\widetilde{a}$, we can make $|\widetilde{a}(u_x)|$ as large as we please uniformly with respect to all $\widetilde{a} \in \widetilde{V}^0_d(F')$ and all finite extensions F'/F by choosing $d \gg 0$ to make $|y(u_x)|^{-d}$ large.

It remains to explain why the largeness condition on d to make $|y(u_x)^{dp}t_x| \leq |\mathbf{q}|$ for an integral formal parameter y as above can be chosen to only depend on g and $\deg_{u,T} h$. Recall that we chose the lift $(\overline{\mathscr{C}}, \widetilde{\xi})$ of $(\overline{C}, \xi)$ so that there exists a lift $H \in \mathscr{A}[T]$ of h such that $\deg_T H = \deg_T h$, $\deg_{u,T} H \leq \deg_{u,T} h + 2g$, and the Laurent expansion of $\mathrm{lead}_T H$ along the W-point $\widetilde{\xi}$ has unit leading coefficient. (This summarizes the essential properties of working with an admissible triple $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ in the sense of Definition 4.4.) These properties, especially the unit condition on $\mathrm{lead}_T H \in W[\![y]\!]$ and the relation $H(y(u_x), t_x) = 0$ with $\deg_T H = \deg_T h > 0$, permit us to use an elementary integrality argument to show that $|y(u_x)^d t_x| \leq 1$ for d large in a sense determined by $\deg_{u,T} h$ and g. Thus, we only have to prove $|y(u_x)^{d(p-1)}| \leq p^{-3}$ for d large in a sense depending only on $\deg_{u,T} h$ and g when $W = W(k)$. But $y(u_x) \in F(x)$ with $|y(u_x)| < 1$, so it is enough to bound $[F(x) : F]$ in terms of $\deg_{u,T} h$ and g. Since $\deg_{u,T} H$ is bounded in terms of $\deg_{u,T} h$ and g, and $[F(x) : F] \leq \deg \mathscr{B}_F$, the problem comes down to bounding the degree of the branch scheme $\mathscr{B}_F$ of the projection $Z_H \to \mathbf{A}^1_F$ in terms of g and the total degree $\deg_{u,T} H$ of H. Such a bound is easily obtained by intersection theory on $\overline{\mathscr{C}}_F \times \mathbf{P}^1_F$, as we explained at the end of the proof of Lemma 2.1. ■

When $|u_x| \leq 1$ and $|t_x| > 1$, we have

$$P^{\mathrm{an}}_{x,d}(\widetilde{a}) = \mathrm{N}_{F(x)/F}(t_x)\mathrm{N}_{F'\otimes_F F(x)/F'}(t_x^{-1}\widetilde{a}(u_x) - 1) \tag{5.12}$$

for F-affinoid algebras F' and $\widetilde{a} \in \widetilde{V}_d(F')$. We absorb the nonzero constant $\mathrm{N}_{F(x)/F}(t_x)$ into b_d in (5.6) for such x's, and the remaining part on the right side of (5.12) is the Raynaud generic fiber of the formal completion of the W-morphism $\underline{\mathscr{V}}_d \to \mathbf{A}^1_W$ defined by

$$\mathrm{N}_{x,d,W} : \widetilde{a} \mapsto \mathrm{N}_{W'\otimes_W W(x)/W'}(t_x^{-1}\widetilde{a}(u_x) - 1)$$

for $\widetilde{a} \in \underline{\mathscr{V}}_d(W')$ with W' a W-algebra. By computing on the generic fiber of the W-flat universal case (*i.e.*, taking $\widetilde{a}$ to be the identity Yoneda-point of the affine W-scheme $\underline{\mathscr{V}}_d$) one sees that the behavior of $\mathrm{N}_{x,d,W}$ under base change on F is just like the behavior of $P_{x,d}$ under such base change. We have therefore almost proved the following result that gives an integral variant on the identity (5.6) when $p = 2$, and it specializes to [3, (5.17)] in the genus-zero case (when we use $\mathscr{A} = W[u]$ and $\widetilde{\underline{\varepsilon}} = \{u^{i-1}\}_{i\geq 1}$):

Theorem 5.6. *Assume $p = 2$. If d is sufficiently large (determined by g and $\deg_{u,T} h$) then there exists $\beta_d = \beta_{d,\widetilde{\underline{\varepsilon}}} \in W^\times$ and a map $\mathbf{U}_d = \mathbf{U}_{d,\widetilde{\underline{\varepsilon}}} : \underline{\mathscr{V}}_d^{0,\wedge} \to \widehat{\mathbf{A}}^1_W$ of formal W-schemes such that*

- *$\mathbf{U}_d$ has nonzero reduction modulo $\mathfrak{m}_W$,*
- *$\mathbf{U}_d^{\mathrm{rig}}$ is a unit on $\widetilde{V}_d^0 = (\underline{\mathscr{V}}_d^{0,\wedge})^{\mathrm{rig}}$,*
- *for all finite extensions F'/F, with valuation ring W' and residue field k', the congruence class $\mathrm{disc}_{\widetilde{\underline{\varepsilon}},\rho(2d)}(H(\widetilde{a}^2)) \bmod 4\mathfrak{m}_W$ is equal to*

$$\beta_d \cdot (1+4s_2(\omega_{h,a})) \cdot \mathbf{U}_d(\widetilde{a})^2 \cdot \prod_{x \in \mathscr{B}_F^{\mathrm{f}}} (P_{x,2d,W})^{e_x}(\widetilde{a}^2) \cdot \prod_{|u_x|\leq 1, |t_x|>1} \mathrm{N}_{x,2d,W}(\widetilde{a}^2)^{e_x} \bmod 4\mathfrak{m}_W \tag{5.13}$$

when $\widetilde{a} \in \widetilde{V}_d^0(F') = \underline{\mathscr{V}}_d^0(W')$ lifts an $a \in \underline{V}_d^0(k')$ for which $h(a^2) \in k' \otimes_k A$ is squarefree; $s_2(\omega_{h,a})$ is given in Definition 5.4.

Moreover, if W is absolutely unramified then the unit residue class $\beta_d \bmod (W^\times)^2$ is independent of the choices of $\mathbf{U}_{d,\widetilde{\underline{\varepsilon}}}$ and of $\widetilde{\underline{\varepsilon}}$.

For any $\widetilde{a} \in \widetilde{V}_d^0(F') = \underline{\mathscr{V}}_d^0(W')$ all factors in (5.13) are in W', so the reduction $a \in \underline{V}_d^0(k')$ yields squarefree $h(a^2)$ in A if and only if all such factors are in $(W')^\times$.

Proof. By applying the congruence $(1-pr)(1+p^2s) \equiv 1 - pr + p^2 s \bmod p^2\mathfrak{m}_W$ to the right side of (5.7) and noting that $\widetilde{c}_{2d+1-g}(\widetilde{a}^2)/\widetilde{c}_{d+1-g}(\widetilde{a})^2$ is a nonzero constant in F that is independent of $\widetilde{a}$ (but depends on $\widetilde{\underline{\varepsilon}}$), the preceding considerations give the desired identity up to the denominator-chasing that ensures that the element $\beta_d \in F^\times$ really lies in $W^\times$. To prove that β_d is an integral unit, it is only necessary to check that all other (formal) functions of $\widetilde{a} \in \underline{\mathscr{V}}_d^{0,\wedge}$ under consideration have *nonzero* reduction modulo $\mathfrak{m}_W$. This is obvious for all functions in (5.13), and the reduction $\mathrm{disc}_{\underline{\varepsilon},\rho(pd)} \circ h((\cdot)^p)$ of $\mathrm{disc}^\wedge_{\widetilde{\underline{\varepsilon}},pd} \circ H^\wedge((\cdot)^p)$ is nonzero on $\underline{V}_d^0$ for large d since (for large d) there exists $a \in \overline{k} \otimes_k A$ with an order-d pole at ξ such that $h(a^p)$ is squarefree in $\overline{k} \otimes_k A$.

Finally, we address the intrinsic nature of the unit residue class $\beta_d \bmod (W^\times)^2$ in the absolutely unramified case. In this case we have $1 + 4\mathfrak{m}_W = 1 + 8W \subseteq (W^\times)^2$. We consider d so large that there exists $a \in \underline{V}_d^0(k)$ for which $h(a^2)$ is squarefree, in which case we may pick a lift $\widetilde{a} \in \underline{\mathscr{V}}_d^0(W)$ such that $H(\widetilde{a}^2)$ has unit discriminant (for any choice of $\widetilde{\underline{\varepsilon}}$). The only property we require for the unit β_d is the congruence formula in (5.13), and knowing β_d modulo $4\mathfrak{m}_W$ certainly determines it modulo $(W^\times)^2$. Since changing $\widetilde{\underline{\varepsilon}}$ changes the discriminant by a unit square algebraic function, and working modulo unit squares eliminates the intervention of the unit square $\mathbf{U}_d(\widetilde{a})^2$, we get the desired result. ∎

6. Quasi-periodicity in characteristic 2

In this final section, we shall use Theorem 5.6 to the study periodicity properties of $\mu(h(a^2))$ for $a \in A$ when $k = \kappa$ is *finite* with characteristic $p = 2$ and $h \in A[T]$ is primitive over A with $h(T^2)$ squarefree in $K[T]$. We also consider questions concerning asymptotics and non-triviality of a correction factor as in §3.

Theorem 4.1 and the inclusion $1+4\mathfrak{m}_W \subseteq (W^\times)^2$ for absolutely unramified W essentially reduce us to understanding the quadratic character of (5.13) when it is a unit and $W = W(\kappa)$. In such cases, the mysterious unit-square contribution $\mathbf{U}_d(\widetilde{a})^2$ can be ignored and (by Theorem 5.6) β_d modulo unit squares is independent of all choices once the admissible lift $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ is selected.

Our treatment of the case of non-unit discriminants rests on a general lemma that has nothing to do with the restriction to residue characteristic 2. Thus, now let W be a mixed-characteristic $(0,p)$ complete discrete valuation ring with perfect residue field k and fraction field F. Using notation as in §5, we prove:

Lemma 6.1. *For all sufficiently large d (determined by g and $\deg_{u,T} h$) and for any $\widetilde{a} \in \underline{\mathscr{V}}_d^0(W)$, if $\mathrm{disc}_{\widetilde{\Xi},\rho(pd)}(H(\widetilde{a}^p)) \in W$ is a non-unit then for some $x \in \mathscr{B}_F^{\mathrm{f}}$ the element $P_{x,pd,W}(\widetilde{a}^p) \in W$ is a non-unit.*

Proof. We can make a finite (possibly ramified) base change on F to reduce to the case when all $x \in \mathscr{B}_F$ are F-rational. Thus, for $x \in \mathscr{B}_F^{\mathrm{f}}$ with reduction $\overline{x} \in B$, $P_{x,\delta,W}$ has reduction $P_{\overline{x},\delta}$ for all large δ. The finite flat W-algebra $\mathscr{A}/(H(\widetilde{a}^p))$ is non-étale over W (*i.e.*, has non-unit discriminant) if and only if its reduction $A/(h(a^p))$ is non-étale over k. By Theorem 4.5, the branch scheme of $Z_h \subseteq C \times \mathbf{A}_k^1$ is the closed fiber of the finite flat W-scheme $\mathscr{B}^{\mathrm{f}}$. Thus, each point of this closed-fiber branch scheme is the reduction of some point in the generic fiber $\mathscr{B}_F^{\mathrm{f}}$. It therefore remains to check that $h(a^p)$ is not squarefree if and only if $P_{\overline{x},pd}(a^p) = 0$ for some branch point $\overline{x}$ of the generically étale projection from $Z_h \subseteq C \times \mathbf{A}_k^1$ to $\mathbf{A}_k^1$. This follows from [5, Thm. 2.6]. ■

Now assume $k = \kappa$ is finite with characteristic $p = 2$ and let $W = W(\kappa)$. Fix an admissible lifting $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ of $(\overline{C}, \xi, h)$ over W in the sense of Definition 4.4. Define $\chi : \kappa^\times \times (1+4W) \twoheadrightarrow \{\pm 1\}$ as in Theorem 4.1, and define $\chi(\mathfrak{m}_W) = \{0\}$. Since $\dim_k A/(h(a^2)) = -\mathrm{ord}_\xi(h(a^2)) \equiv -\mathrm{ord}_\xi(\mathrm{lead}(h)) \bmod 2$ when $-\mathrm{ord}_\xi(a) \gg 0$, by using Theorem 4.1, Theorem 5.6, and Lemma 6.1 we conclude:

Theorem 6.2. *For sufficiently large d and any $a \in \underline{V}_d^0(\kappa)$, $\mu(h(a^2))$ is given by the formula*
(6.1)

$$(-1)^{d_0+\mathrm{Tr}_{\kappa/\mathbf{F}_2}(s_2(\omega_{h,a}))}\chi\left(\beta_d \prod_{x\in\mathscr{B}_F^{\mathrm{f}}} P_{x,2d,W}(\widetilde{a}^2)^{e_x} \cdot \prod_{|u_x|\le 1, |t_x|>1} \mathrm{N}_{W(x)/W}(t_x^{-1}\widetilde{a}(u_x)^2 - 1)^{e_x}\right),$$

where $d_0 = -\mathrm{ord}_\xi(\mathrm{lead}(h))$, $\widetilde{a} \in \underline{\mathscr{V}}_d^0(W)$ lifts a, and $s_2(\omega_{h,a})$ is given in Definition 5.4. The largeness of d is determined by g and $\deg_{u,T} h$.

Implicit in this theorem is the fact (immediate from Remark 4.2, (5.13), and Lemma 6.1) that if the quantity on which we are evaluating χ in (6.1) is a unit then its 1-unit part lies in $1 + 4W$. Observe that the only ingredient "inside" of χ in (6.1) that depends on $d = -\mathrm{ord}_\xi(a)$ is β_d, since $P_{x,2d,W}$ is a norm-evaluation construction having no dependence on pole-orders at ξ (the subscript $2d$ merely indicates that we are evaluating on points $\widetilde{a}^2$ of $\underline{\mathscr{V}}_{2d}$).

Corollary 6.3. *For all $d, d' \gg 0$ (only depending on g and $\deg_{u,T} h$), $\beta_d/\beta_{d'} \in W^\times$ lies in $\kappa^\times \times (1+4W)$.*

This corollary ensures that for large d and d' it makes sense to compute $\chi(\beta_d/\beta_{d'})$.

Proof. To deduce properties of β_d as d varies through large values as above, pick $d \gg 0$ and use Lemma 2.1 to find $a \in \underline{V}_d^0(\kappa)$ with $h(a^2)$ squarefree. Let $\mathscr{Y} \subseteq \overline{\mathscr{C}}$ be the closure of the union of the closed points $u_x \in \mathscr{C}_F$ with $|u_x| \leq 1$ for $x \in \mathscr{B}_F$, so $\mathscr{Y}$ is a relative effective Cartier divisor and its reduction $\mathscr{Y}_\kappa \subseteq \overline{C}$ is supported in $C = \operatorname{Spec} A$ with κ-degree equal to $\deg(\mathscr{Y}_F) = \sum_{|u_x|\leq 1}[F(u_x) : F]$. For any large d' (only depending on the genus of $\overline{C}$ and $\deg(\mathscr{Y})$) we can find $a' \in A$ with a pole of order d' at ξ and with values along $\mathscr{Y}_\kappa$ equal to those of a. Hence, for any $\widetilde{a} \in \underline{\mathscr{V}}_d^0(W)$ and $\widetilde{a}' \in \underline{\mathscr{V}}_{d'}^0(W)$ lifting a and a' respectively, we are assured that the values $\widetilde{a}(u_x)$ and $\widetilde{a}'(u_x)$ in the valuation ring $W(u_x)$ of $F(u_x)$ have the same image in the residue field of $W(u_x)$ whenever $|u_x| \leq 1$. (Such lifts $\widetilde{a}$ and $\widetilde{a}'$ can always be found provided that d and d' are large enough in a manner that is determined by the genus of the curve.)

Since $\mu(h(a^2)) \neq 0$, it follows from (6.1) and the congruence of u_x-values that $\mu(h({a'}^2)) \neq 0$. The congruence of $\widetilde{a}(u_x)$ and $\widetilde{a}'(u_x)$ modulo $\mathfrak{m}_{W(u_x)}$ whenever $|u_x| \leq 1$ implies that $\widetilde{a}^2(u_x) \equiv (\widetilde{a}')^2(u_x) \bmod 2\mathfrak{m}_{W(u_x)}$ for such u_x. Thus, by Galois-invariance the two products in (6.1) for $\widetilde{a}$ and $\widetilde{a}'$ are congruent modulo $2\mathfrak{m}_W = 4W$. However, the 1-unit part of the expression "inside" of χ in (6.1) for $\widetilde{a}$ and $\widetilde{a}'$ both lie in $1+4W$, so taking ratios gives that the W-unit $\beta_d/\beta_{d'}$ has 1-unit part in $1+4W$. ■

Once the choice of admissible lift $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ over $W = W(\kappa)$ is fixed, the e_x's do not depend on any further non-canonical choices (such as $\underline{\varepsilon}$), and likewise the unit $\beta_d \in W^\times$ taken modulo $(W^\times)^2$ for $d \gg 0$ is also independent of non-canonical choices. In particular, $\chi(\beta_d/\beta_{d'})$ is independent of the choices of $\underline{\varepsilon}$ and $\widetilde{\underline{\varepsilon}}$. However, the β_d's (as well as $P_{x,2d,W}$ and $\mathscr{B}$) depend on $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$.

Motivated by (6.1), we now use $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ to define a nonzero (possibly non-radical) ideal in A that will play the role for characteristic 2 that the radical ideal I_f as in §1 played in our considerations in odd characteristic in [5].

Definition 6.4. Define the ideal $\mathscr{I}_H \subseteq \mathscr{A}$ to be the radical ideal such that $\operatorname{Spec}(\mathscr{A}/\mathscr{I}_H) \subseteq \mathscr{C}$ has support equal to the union of the closed subschemes $\{u_x\}$ in $\mathscr{C}$ as x ranges over points of $\mathscr{B}_F$ such that $|u_x| \leq 1$. Equivalently, $\operatorname{Spec}\mathscr{A}/\mathscr{I}_H$ is the W-finite flat reduced closed subscheme of $\mathscr{C}$ obtained by forming the closure in $\overline{\mathscr{C}}$ of the reduced divisor on $\mathscr{C}_F$ supported at the $u_x \in \mathscr{C}_F$ whose reduction in $\overline{C}$ is not ξ.

Define the nonzero ideal $I_H \subseteq A = \mathscr{A}/\mathfrak{m}_W\mathscr{A}$ to be the reduction $(\mathscr{I}_H + \mathfrak{m}_W\mathscr{A})/\mathfrak{m}_W\mathscr{A} \simeq \mathscr{I}_H/\mathfrak{m}_W\mathscr{I}_H$ of $\mathscr{I}_H$, so $\operatorname{Spec} A/I_H \subseteq C$ is the closed fiber of the W-flat $\operatorname{Spec}\mathscr{A}/\mathscr{I}_H \subseteq \mathscr{C}$. Define $I_{f,\kappa} \subseteq A$ to be the gcd of the I_H's as we vary over all admissible triples $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ over $W(\kappa)$ lifting $(\overline{C}, \xi, h)$.

The ideal I_H depends on the admissible triple $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$, as does the ideal $\mathscr{I}_H$, but we prefer to emphasize just the dependence on H in the notation. Note that I_H need not be a radical ideal (an explicit non-radical example is given in the case $A = \kappa[u]$ in [3, Ex. 5.15], with $\mathscr{I}_H$ and I_H equal to the principal ideals generated by $(M_H^{\text{geom}})^{\leq 1}$ and $\overline{M}_H^{\text{geom}}$ in $W[u]$ and $\kappa[u]$ respectively). By Theorem 6.2, the property of $\mu(h(a^2))$ vanishing or not is determined by $a \bmod I_H$ for $-\operatorname{ord}_\xi(a) \gg 0$, with largeness that is determined by g and $\deg_{u,T} h$.

The W-algebra $\mathscr{A}/\mathscr{I}_H$ is finite and flat, and the two products in (6.1) define a map of sets $\mathscr{A}/\mathscr{I}_H \to W$ via

$$\begin{aligned}\mathfrak{L}_W : \widetilde{a} \quad &\mapsto \quad \prod_{x \in \mathscr{B}^{\mathrm{f}}_F} \mathrm{N}_{W(x)/W}(\widetilde{a}(u_x)^2 - t_x)^{e_x} \cdot \prod_{|u_x| \leq 1, |t_x| > 1} \mathrm{N}_{W(x)/W}(t_x^{-1}\widetilde{a}(u_x)^2 - 1)^{e_x} \\ &= \quad \mathrm{N}_{\mathscr{B}^{\mathrm{f}}/W}(\widetilde{a}^2 - T) \cdot \prod_{|u_x| \leq 1, |t_x| > 1} \mathrm{N}_{W(x)/W}(t_x^{-1}\widetilde{a}(u_x)^2 - 1)^{e_x}.\end{aligned}$$

The same formula makes sense after the finite étale extension of scalars $W = W(\kappa) \to W(\kappa') = W'$ for any finite extension κ'/κ; we simply have to take into account that each of the x's may decompose in several points (and this is compatible with formation of the associated valuation rings since the base change $W \to W'$ is finite étale). Likewise, for $n \geq 1$ we get maps modulo 2^n given by the same formula, and if $n > 1$ then when working modulo 2^n it suffices to take the source modulo 2^{n-1} since squaring promotes a congruence modulo 2^{n-1} to a congruence modulo 2^n for $n > 1$.

We need to precisely formulate the fact that these maps modulo 2^n are algebraic in the varying κ'. For $n \geq 1$ and a finite flat $W_n(\kappa)$-algebra $\mathscr{S}$, the functor $W_{\mathscr{S},n} : R \rightsquigarrow W_n(R) \otimes_{W_n(\kappa)} \mathscr{S}$ on κ-algebras is represented by a ring scheme over κ whose underlying κ-scheme is an affine space; here and below, W_n denotes the functor of 2-adic truncated Witt vectors $(w_0, \dots, w_{n-1})$ of length n. Let $\mathscr{R}_n$ denote the ring scheme representing $W_{\mathscr{S},n}$ in the case $\mathscr{S} = \mathscr{A}/(2^n, \mathscr{I}_H)$, so for $n > 1$ we may using the "squaring" morphism $\mathscr{R}_{n-1} \to \mathscr{R}_n$ to obtain a unique map of κ-schemes $\mathfrak{L}_n : \mathscr{R}_{n-1} \to W_n$ whose induced map on κ'-points is

$$\mathfrak{L}_{W(\kappa')} \bmod 2^n : W_{n-1}(\kappa') \otimes_{W_{n-1}(\kappa)} (\mathscr{A}/\mathscr{I}_H) = (W' \otimes_W \mathscr{A})/(2^{n-1}, \mathscr{I}_H) \to W'/2^n W'$$

for varying finite extensions κ'/κ.

By Theorem 6.2 and the existence of squarefree values $h(a^2)$ for a with sufficiently large ξ-degree, the $\mathfrak{L}_3$-preimage $U' \subseteq \mathscr{R}_2$ of the Zariski-dense open subvariety of units $W_3^\times \subseteq W_3$ is nonempty (hence Zariski-dense in the affine space $\mathscr{R}_2$). Moreover, as we noted after Theorem 6.2, for all large d (and fixed $\widetilde{\varepsilon}$) the product map $\beta_d \mathfrak{L}_3|_{U'} : U' \to W_3^\times$ against $\beta_d = \beta_{d,\widetilde{\varepsilon}}$ is valued in the subgroup $\mathbf{G}_m \times \mathbf{G}_a$ of units with vanishing middle component: $(u, 0, u^4 z) = [u] \cdot (1, 0, z)$ (with $[\cdot]$ denoting the Teichmüller section $\mathbf{G}_m \to W_3^\times$).

Our primary interest will be the case when h is a polynomial in T^2 (i.e., $f = h(T^2)$ lies in $A[T^4]$), so consider the composition of $\mathfrak{L}_3$ with the "squaring map" $\mathscr{R}_1 \to \mathscr{R}_2$. By the same method as above, this composite hits $W_3^\times$ and so on the Zariski-dense open preimage $U \subseteq \mathscr{R}_1$ of $W_3^\times$ we get a κ-map $\mathfrak{L} : U \to W_3^\times$. For any point $u = (u_0, \dots)$ of the unit group functor $W_n^\times$ we write $\langle u \rangle$ to denote the 1-unit factor $[u_0]^{-1} u$, and for any point w of the ring functor W_n we write w_i to denote the ith coordinate of this truncated Witt vector for $0 \leq i \leq n - 1$.

For large d, since the product $\beta_d \mathfrak{L} : U \to W_3^\times$ has its 1-unit factor lying in $\mathbf{G}_a = \{(1, 0, z)\}$, the formation of this 1-unit gives an algebraic map of κ-varieties $\langle \beta_d \mathfrak{L} \rangle_2 : U \to \mathbf{A}^1_\kappa$. We emphasize that U is a dense open in the ring scheme $\mathscr{R}_1$ associated to the finite κ-algebra A/I_H. By Corollary 6.3, for large d and d' the ratio $\beta_d/\beta_{d'} \bmod 8$ has 1-unit component of the form $(1, 0, z_{d,d'})$ for some $z_{d,d'} \in \kappa$ since we are assuming h is a polynomial in T^2; Corollary 6.12 below ensures that we can take $z_{d,d'} = 0$ when d and d' have the same parity. In particular, the isomorphism class of the double cover of U defined by $y^2 - y = \langle \beta_d \mathfrak{L} \rangle_2$ only depends on $d \bmod 2$ for large d.

We conclude that the two maps

$$\langle \beta_d \mathfrak{L} \rangle_2, \langle \beta_{d'} \mathfrak{L} \rangle_2 : U \rightrightarrows \mathbf{A}^1_\kappa$$

are related through additive translation by $z_{d,d'}$, and by (6.1) the study of $\mu(h(a^2))$ for $h \in A[T^2]$ and sufficiently large $d = -\mathrm{ord}_\xi(a)$ is governed by the Artin–Schreier covering of U defined by $y^2 - y = \langle \beta_d \mathfrak{L} \rangle_2$ because (i) $h(a^2)$ is squarefree if and only if a mod I_H lies in $U(\kappa)$ inside of A/I_H, and (ii) for any finite extension κ'/κ, a point $w = (u, 0, z) \in W_3(\kappa')^\times$ is a square if and only if its 1-unit coordinate $\langle w \rangle_2 = u^{-4}z \in \kappa'$ has vanishing trace into $\mathbf{F}_2$, or equivalently $u^{-4}z$ has the form $y^2 - y$ for some $y \in \kappa'$.

Remark 6.5. Fix $f \in A[T^p]$ as in §3 with $p \neq 2$. In our study of Möbius averages for f in §3, we saw that the asymptotic behavior of the periodic sequence of such averages across large degrees for increasing constant fields (with f fixed) is controlled by a degree-2 Kummer covering of a dense open in an affine space (via extraction of a square root of $\prod_{x \in B} P_x^{\ell(\mathscr{O}_{B,x})}$). An analogous result will be proved in Theorem 6.14 for $p = 2$ and $f \in A[T^4]$ by working with Artin–Schreier double covers as above.

Returning now to the general case with $h \in A[T]$ for which $h(T^2) \in K[T]$ is squarefree (but perhaps h is not in $A[T^2]$), we have seen that congruence modulo the ideal I_H in A for any choice of H as above controls whether or not $h(a^2)$ is squarefree in A. It is reasonable to ask if we can construct a nonzero multiple of I_H without appealing to lifts into characteristic 0. In the "generic" case when $\mathrm{lead}(h) \in A$ has no double zeros (*e.g.*, when h is monic), we can control some properties of I_H by using equicharacteristic geometric constructions; in particular, we can bound I_H by equicharacteristic methods when $\mathrm{lead}(h)$ has no double zeros. This is the content of the next theorem.

Theorem 6.6. *Let $B \subseteq Z = Z_h$ be the κ-finite branch scheme of the projection $Z \to \mathbf{A}^1_\kappa$. Define the nonzero ideal $\mathrm{Fitt}_{B/C} \subseteq A$ to be the Fitting ideal of $\mathscr{O}_B$ viewed as a finite-length A-module. Assume that $\mathrm{lead}(h) \in A$ has no double zeros on C. The ideal I_H divides the Fitting ideal $\mathrm{Fitt}_{B/C}$, and these have the same radical, with zero locus equal to the image of B in C.*

This theorem specializes to [3, Lemma 5.9] in the genus-0 case, and by [3, Ex. 5.15] the irreducible factors of I_H may occur with higher multiplicity in $\mathrm{Fitt}_{B/C}$ than they do in I_H.

Proof. To establish equality of zero loci, we just have to show that if $x \in \mathscr{B}_F$ satisfies $|u_x| \leq 1$ then necessarily $|t_x| \leq 1$ (so $x \in \mathscr{B}_F^{\mathrm{f}}$). Since $H(u_x, t_x) = 0$, if $\mathrm{lead}(H)$ has integral unit value at u_x then certainly $|t_x| \leq 1$. However, since $\mathscr{A}/(\mathrm{lead}(H))$ is finite flat of degree d_0 over W (because $\mathrm{lead}(H) \in \underline{\mathscr{V}}^0_{d_0}(W)$ by construction of $(\overline{\mathscr{C}}, H, \widetilde{\xi})$) and its reduction $A/(\mathrm{lead}(h))$ is finite étale over κ, it follows that $\mathscr{A}/(\mathrm{lead}(H))$ is finite étale over W. Since $\Omega^1_{\mathscr{A}/W}$ is an invertible $\mathscr{A}$-module with reduction $\Omega^1_{A/\kappa}$ as an A-module, we can find a W-linear derivation $D : \mathscr{A} \to \mathscr{A}$ that is dual to a local generator of $\Omega^1_{\mathscr{A}/W}$ near the zeros of $\mathrm{lead}(h)$ on C.

Assuming $\mathrm{lead}(H)$ has value at u_x that is not an integral unit, it follows from the "simple zeros" hypothesis and the condition $|u_x| \leq 1$ that $D(\mathrm{lead}(H))(u_x)$ must be an integral unit. Since the point (u_x, t_x) in the branch scheme $\mathscr{B}$ of $\mathscr{Z}(H) \to \mathbf{A}^1_W$ must lie in the zero locus of DH, the resulting equation $(DH)(u_x, t_x) = 0$ is a polynomial relation for t_x with integral coefficients and unit leading coefficient. Hence, once again $|t_x| \leq 1$. This proves that I_H

and $\mathrm{Fitt}_{B/C}$ have the same zero locus on C, namely the image of B in C (*i.e.*, the support of $\mathscr{O}_B$ as an A-module).

To prove that $I_H|\mathrm{Fitt}_{B/C}$, recall that I_H is the reduction of $\mathscr{I}_H$, with $\mathscr{A}/\mathscr{I}_H$ flat over W. Also, by Theorem 4.5, $\mathrm{Fitt}_{B/C}$ is the reduction of the Fitting ideal of the W-finite $\mathscr{O}_{\mathscr{B}^{\mathrm{f}}}$ over $\mathscr{A}$. The inclusion $\mathrm{Fitt}_{B/C} \subseteq I_H$ holds if and only if $\mathrm{Fitt}_{B/C}$ has vanishing image in A/I_H, so it suffices to show that $\mathrm{Fitt}_{\mathscr{A}}(\mathscr{O}_{\mathscr{B}^{\mathrm{f}}})$ vanishes in $\mathscr{A}/\mathscr{I}_H$. This latter quotient is W-flat, so it suffices to check the vanishing on generic fibers over W. That is, it suffices to show that the Fitting ideal of $\mathscr{B}^{\mathrm{f}}_F$ as a finite-length $\mathscr{A}_F$-module is divisible by the generic-fiber ideal $F \otimes_W \mathscr{I}_H$. This latter ideal is the radical ideal with zero-locus equal to the image of $\mathscr{B}^{\mathrm{f}}_F$ under projection to $\mathscr{C}_F$ (since $|u_x| \le 1$ forces $|t_x| \le 1$, as we saw above), so we are reduced to showing that the image of the projection $\mathscr{B}^{\mathrm{f}}_F \to \mathscr{C}_F$ is contained in the zero locus of the Fitting ideal of $\mathscr{O}_{\mathscr{B}^{\mathrm{f}}_F}$ as an $\mathscr{A}_F$-module. The zero-locus of the Fitting ideal of a finitely presented module is equal to the support of the module, so we just have to observe that the support of $\mathscr{O}_{\mathscr{B}^{\mathrm{f}}_F}$ as a finite(-length) $\mathscr{A}_F$-module is obviously equal to the image of $\mathscr{B}^{\mathrm{f}}_F$ under projection to $\mathscr{C}_F = \mathrm{Spec}\,\mathscr{A}_F$. ■

Now suppose (with $p = 2$) that $f \in A[T^4]$ is primitive with respect to A and squarefree in $K[T]$ with positive degree. Write $f = h(T^2)$ with $h \in A[T^2]$, so by [5, Thm. 2.6] the quasi-finite projection $Z_h \to \mathbf{A}^1$ is generically étale. Let $I_h \subseteq A$ be the nonzero radical ideal whose zero locus is the image in C of the finite branch scheme for projection from Z_h to $\mathbf{A}^1$. By Theorem 6.6, under a mild hypothesis on f the ideal I_h is useful in the study of Möbius periodicity:

Corollary 6.7. *With notation and hypotheses as above, assume that the leading coefficient* $\mathrm{lead}(f) = \mathrm{lead}(h)$ *in A has no double zeros on C. The radical of the ideal $I_{f,\kappa}$ in Definition* 6.4 *is equal to the radical ideal I_h.*

We are now in position to state the main periodicity theorem in characteristic 2, analogous to [5, Thm. 1.2] for $p \neq 2$. The genus-0 case was treated in [3, Thm. 5.12], and this special case will be used in the proof of the general case via essentially the same projection technique that we used in odd characteristic in [5, §6] to relate higher genus and genus 0.

Theorem 6.8. *Assume $k = \kappa$ is finite of characteristic* 2, *and let $f \in A[T^2]$ be primitive with $f \in K[T]$ squarefree of positive degree. Write $f = h(T^2)$. Choose an admissible lifting $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ of $(\overline{C}, \xi, h)$ over $W = W(\kappa)$, and define the nonzero ideal $I_H \subseteq A$ as in Definition* 6.4. *For any $a \in A$, define the meromorphic* 1*-form $\omega_{h,a} = ((\partial_T h)(a^2)a/h(a^2))\,\mathrm{d}a$ on $\overline{C}$, and define $s_2(\omega_{h,a}) \in \kappa$ to be the second symmetric function of its residues, indexed by the geometric poles.*

Consider the function $\widetilde{\mu}_f : a \mapsto (-1)^{\mathrm{Tr}_{\kappa/\mathbf{F}_2}(s_2(\omega_{h,a}))}\mu(f(a))$ on A. If $a, a' \in A$ are nonzero then

$$a \equiv a' \bmod I_H,\ \ \mathrm{ord}_\xi(a) \equiv \mathrm{ord}_\xi(a') \bmod 4 \Rightarrow \widetilde{\mu}_f(a) = \widetilde{\mu}_f(a') \tag{6.2}$$

provided $-\mathrm{ord}_\xi(a), -\mathrm{ord}_\xi(a') \gg 0$ *(with largeness determined by g and $\deg_{u,T} f$).*

If $\deg h$ is even then the congruence on ord_ξ's in (6.2) *need only be taken modulo* 2, *and if $[\kappa : \mathbf{F}_2]$ is even or $4|\deg h$ then this congruence condition can be dropped.*

The congruence between a and a' in (6.2) may even be taken modulo the ideal $I_{f,\kappa}$ that is the gcd of the nonzero ideals I_H as we vary over all admissible lifts $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ over $W(\kappa)$. This is an easy consequence of Theorem 6.8, using the Chinese remainder theorem and Riemann–Roch. Before we begin the proof of Theorem 6.8, we record the following immediate corollary concerning Möbius periodicity in characteristic 2.

Corollary 6.9. *Assume $k = \kappa$ is finite of characteristic 2, and let $f \in A[T^4]$ be primitive with $f \in K[T]$ squarefree of positive degree. Define the nonzero ideal $I_{f,\kappa} \subseteq A$ as in Definition 6.4. For $a, a' \in A$ with sufficiently large pole orders at ξ,*

$$a \equiv a' \bmod I_{f,\kappa}, \;\; \mathrm{ord}_\xi(a) \equiv \mathrm{ord}_\xi(a') \bmod 2 \Rightarrow \mu(f(a)) = \mu(f(a')).$$

The congruence condition modulo 2 may be dropped if $[\kappa : \mathbf{F}_2]$ is even or $\deg_T f$ is divisible by 8. The "sufficiently largeness" of pole-orders at ξ is determined by g and $\deg_{u,T} f$.

The proof of Theorem 6.8 requires a standard cohomological result that we recall for ease of reference (and whose proof we omit):

Lemma 6.10. *Let $X \to S$ be a proper smooth map with geometrically connected fibers of dimension 1, with S local and s its closed point. Let $D_1, D_2 \subseteq X$ be relative effective Cartier divisors. For sufficiently large d only depending on the $\deg D_j$'s and the genus of X_s, the natural map $\mathrm{H}^0(X, \mathscr{O}_X(d \cdot D_1 - D_2)) \to \mathrm{H}^0(X_s, \mathscr{O}_{X_s}(d \cdot (D_1)_s - (D_2)_s))$ is surjective.*

Proof. (of Theorem 6.8). The proof is long, so we break it up into several steps.

Step 1. We begin by rephrasing the problem in terms of the quadratic character of $\beta_d/\beta_{d'} \in W^\times$ (as in Corollary 6.3) for large d and d' that are congruent modulo 4. Consider $\mathrm{Spec}\, \mathscr{A}/\mathscr{I}_H$. By definition, this is the schematic closure of an F-finite (reduced) closed subscheme of $\overline{\mathscr{C}}_F$ and it is disjoint from the section $\widetilde{\xi}$, so it is finite and flat over W. In particular, it is a relative effective Cartier divisor on $\overline{\mathscr{C}}$ over W. Thus, Lemma 6.10 ensures that any congruence $a \equiv a' \bmod I_H$ (*i.e.*, equality in the closed fiber A/I_H of $\mathscr{A}/\mathscr{I}_H$) with $a \in \underline{V}^0_d(\kappa)$ and $a' \in \underline{V}^0_{d'}(\kappa)$ may be lifted to a congruence $\widetilde{a} \equiv \widetilde{a}' \bmod \mathscr{I}_H$ with $\widetilde{a} \in \underline{\mathscr{V}}^0_d(W)$ and $\widetilde{a}' \in \underline{\mathscr{V}}^0_{d'}(W)$ when d and d' are sufficiently large (with largeness only depending on g and $\deg_{u,T} f$).

Recall that the only ingredient in the formation of (6.1) that depends on $d = -\mathrm{ord}_\xi(a)$ is β_d, and (as we recorded in Corollary 6.3) the ratios $\beta_d/\beta_{d'} \in W^\times$ lie in $\kappa^\times \times (1 + 4W)$ for $d, d' \gg 0$. We may therefore conclude from the Möbius formula (6.1) that Theorem 6.8 is equivalent to the claim that for $d, d' \gg 0$, (i) $\chi(\beta_d/\beta_{d'}) = 1$ if $d \equiv d' \bmod 4$ and (ii) this congruence condition can be weakened to $d \equiv d' \bmod 2$ when $\deg h$ is even and it can be eliminated (*i.e.*, $\beta_d/\beta_{d'} \in (W^\times)^2$ for all $d, d' \gg 0$) when $[\kappa : \mathbf{F}_2]$ is even or $4 | \deg h$. This dependence can be made very explicit in the genus-0 case, since our explicit formula for $\mu(h(a^2))$ in [3] yields $\chi(\beta_d/\beta_{d'}) = (-1)^{[\kappa:\mathbf{F}_2](\lfloor (1+d\deg h)/2 \rfloor + \lfloor (1+d'\deg h)/2 \rfloor)}$ in the genus-0 case.

To understand how β_d modulo $(W^\times)^2$ depends on d for higher-genus cases, we will use a modification of the method applied in [5, §6] in odd characteristic. Recall that in [5, §6] we used well-chosen projections to the projective line to pull up properties from the genus-0 case. We will need a variant on this method, adapted to the use of lifts to characteristic 0.

Step 2. We want to construct a projection from $\overline{\mathscr{C}}$ to $\mathbf{P}^1_W$ that relativizes the construction over k in [5, §5]. For conceptual clarity, briefly fix a base scheme S_0 (such as $\mathrm{Spec}\, W$) and a proper smooth morphism $\phi : X \to S_0$ with 1-dimensional fibers that are geometrically connected of genus g (*e.g.*, $\overline{\mathscr{C}}$), as well as a section $x \in X(S_0)$. Fix an integer $r \geq 2g - 1$. By the theorem on cohomology and base change, $\mathscr{V}_r = \phi_*(\mathscr{O}(r \cdot x))$ is a vector bundle on S_0 of rank $r + 1 - g$ whose formation commutes with base change, and $\mathscr{V}_{r-1}$ is a codimension-1 subbundle of $\mathscr{V}_r$ for $r \geq 2g$. For $r \geq 2g$, define the functor H_r on S_0-schemes S by letting $H_r(S)$ be the set of finite flat S-maps $\pi : X \times_{S_0} S \to \mathbf{P}^1_S$ of degree r such that $\pi^{-1}(\infty) = r \cdot x_S$

as relative effective Cartier divisors on $X \times_{S_0} S$. This functor is represented by the open subscheme $\underline{\mathscr{V}}_r^0 := \underline{\mathscr{V}}_r - \underline{\mathscr{V}}_{r-1}$ inside of $\underline{\mathscr{V}}_r$, and we shall also write H_r to denote this representing object. The subfunctor H_r^0 classifying those $\pi \in H_r(S)$ such that $\pi_s \in H_r(s)$ is generically étale for all $s \in S$ is represented by an open subscheme in H_r, and this open subscheme is also denoted H_r^0. Note that H_r^0 is smooth over S_0 and is fiberwise nonempty (hence fiberwise dense) in H_r when $r \geq 2g+1$ (see the beginning of [5, §5] for proofs of these claims when $S_0 = \operatorname{Spec} L$ for a field L; the proofs in the general case go the same way).

We apply these considerations to our curve $\overline{\mathscr{C}}$ over $S_0 = \operatorname{Spec} W$ with $x = \widetilde{\xi}$. Fix an odd integer $r \geq 2g+1$, and let $\pi^{\mathrm{univ}} : \overline{\mathscr{C}} \times_{S_0} H_r^0 \to \mathbf{P}^1_{H_r^0}$ be the universal degree-r morphism over the smooth faithfully flat W-scheme H_r^0. By [5, Thm. 5.2], the locus of points s in the closed fiber $(H_r^0)_\kappa$ such that $\mathrm{N}_{\pi_s^{\mathrm{univ}}}(h) \in k(s)[u,T]$ is squarefree is a dense Zariski-open in $(H_r^0)_\kappa$. The scheme $(H_r^0)_\kappa$ is a nonempty open inside of an affine space, so by [5, Lemma 6.1] it must contain closed points over a finite extension of κ that may be chosen with κ-degree relatively prime to any specified nonzero integer. Pick a closed point s_0 in the closed fiber of H_r^0 with $[\kappa(s_0) : \kappa]$ odd. Let $\kappa' = \kappa(s_0)$ and let $W' = W(\kappa')$ be the finite étale local extension of W with residue field κ'. Let $\pi_0 : \overline{C}_{\kappa'} \to \mathbf{P}^1_{\kappa'}$ be the κ'-map corresponding to s_0, and let $\chi' : {\kappa'}^\times \times (1+4W') \twoheadrightarrow \{\pm 1\}$ be the unique quadratic character with kernel $({W'}^\times)^2$. Note that the restriction of χ' to $\kappa^\times \times (1+4W)$ is χ because $\chi(c \cdot (1+4w)) = (-1)^{\mathrm{Tr}_{\kappa/\mathbf{F}_2}(w \bmod 2)}$ and the transitivity of traces yields

$$\mathrm{Tr}_{\kappa'/\mathbf{F}_2}|_\kappa = [\kappa' : \kappa] \cdot \mathrm{Tr}_{\kappa/\mathbf{F}_2} = \mathrm{Tr}_{\kappa/\mathbf{F}_2}$$

since $[\kappa' : \kappa] \equiv 1 \bmod 2$. Thus, by Theorem 4.1 we can rename κ' as κ (so $H_r^0(\kappa)$ is nonempty), and we can make further odd-degree extensions on κ without loss of generality.

Choose integers $d_i > 0$ forming a system of representatives for $\mathbf{Z}/r\mathbf{Z}$ such that there exist $\widetilde{a}_i \in \underline{\mathscr{V}}^0_{d_i}(W)$ with reduction $a_i \in \underline{V}^0_{d_i}(k)$ not vanishing at the zeros of $h(0) \neq 0$; such $\widetilde{a}_i$ can be found for $d_i \gg 0$ (only depending on g) by Riemann–Roch. Define the primitive polynomial $h_i(T) = h(a_i^2 T) \in A[T]$, so $h_i(T^2) = f(a_i T)$ is squarefree in $K[T]$ and $H_i(T) := H(\widetilde{a}_i^2 T) \in \mathscr{A}[T]$ is an admissible lift of $h_i(T)$ (in the sense of Definition 4.4). Since W is henselian, smoothness of H_r^0 over W ensures that $H_r^0(W) \to H_r^0(\kappa)$ is surjective. Thus, by the functorial interpretation of H_r^0 as a Hom-scheme, we may pick a degree-r finite flat map $\pi : \overline{\mathscr{C}} \to \mathbf{P}^1_W$ lifting π_0 such that $\pi^{-1}(\infty) = r \cdot \widetilde{\xi}$ and the open étale locus for π in $\overline{\mathscr{C}}$ is dense in fibers of $\overline{\mathscr{C}}$ over $\operatorname{Spec} W$. Since $h_i(T^2)$ is squarefree and any two nonempty Zariski-opens in an affine space over a field must have nonempty intersection, by [5, Thm. 5.2] over κ we may also suppose (upon making a further odd-degree extension of κ that we promptly rename as κ) that π_0 was chosen so that the π_0-norm of $h_i(T^2)$ down to the affine line $\mathbf{A}^1_\kappa$ is squarefree in $\kappa[u,T]$.

Step 3. Let us now introduce the objects that will enable us to relate the general case with the genus-0 case. Using h_i and H_i as in Step 2, define $\mathbf{h}_i = \mathrm{N}_{\pi_0}(h_i) \in \kappa[u][T]$ (a primitive polynomial in T) and $\mathbf{H}_i = \mathrm{N}_\pi(H_i) \in W[u][T]$, so $\mathbf{h}_i(T^2)$ is squarefree over $\kappa(u)$ and $\mathbf{H}_i$ is a lift of $\mathbf{h}_i$ with the same T-degree such that the respective leading coefficients of $\mathbf{H}_i$ and $\mathbf{h}_i$ in $W[u]$ and $\kappa[u]$ have the same u-degree. Beware that the total degree $\deg_{u,T} \mathbf{H}_i$ may be larger than that of $\deg_{u,T} \mathbf{h}_i$, but this will be unimportant later because what matters is that the possible excess is bounded above depending only on r, d_i, and $\deg_{u,T} h$. (Keep in mind that $r \geq 2g+1$ and the total degree $\deg_{u,T} H$ is bounded above in terms of g and $\deg_{u,T} h$.)

For $a \in A$ and $\varphi \in \kappa[u]$, define the meromorphic 1-forms

$$\omega_{h_i,a} = \frac{(\partial_T h_i)(a^2)a}{h_i(a^2)}\,\mathrm{d}a, \;\; \omega_{\mathbf{h}_i,\varphi} = \frac{(\partial_T \mathbf{h}_i)(\varphi^2)\varphi}{\mathbf{h}_i(\varphi^2)}\,\mathrm{d}\varphi$$

on $\overline{C}$ and $\mathbf{P}^1_\kappa$ respectively. (The denominators in these two definitions are nonzero because $h_i(T^2)$ and $\mathbf{h}_i(T^2)$ are squarefree in the $\mathbf{F}_2$-algebras $K[T]$ and $\kappa(u)[T]$ respectively.) Define the invertible ideal $\mathscr{I}_i \subseteq W[u]$ to be the product of the invertible $\mathscr{I}_{\mathbf{H}_i}$ and the invertible ideal defining the W-finite flat schematic image of the finite map $\operatorname{Spec}\mathscr{A}/\mathscr{I}_{H_i} \to \mathbf{A}^1_W$ induced by π. The quotient $W[u]/\mathscr{I}_i$ is finite and flat over W. Define $I_i \subseteq \kappa[u]$ to be the nonzero reduction of $\mathscr{I}_i$.

Fix i with $1 \le i \le r$. By Lemma 2.1, for $d \gg 0$ (with largeness depending only on the genus, r, and $\deg_{u,T} f$) there exists $\varphi \in \kappa[u]$ of degree d such that $\mathbf{h}_i(\varphi^2) \in \kappa[u]$ is squarefree. Since $\mathbf{h}_i(\varphi^2) = \mathrm{N}_{\pi_0}(h_i(\pi_0^*(\varphi)^2))$, obviously the nonzero $h_i(\pi_0^*(\varphi)^2) \in A$ is squarefree with physical zeros in distinct fibers of π_0. Thus, by the definition of the Möbius function, $\mu(A/(h_i(\pi_0^*(\varphi)^2))) = \mu(\kappa[u]/(\mathbf{h}_i(\varphi^2)))$. Since the desired implication (6.2) involves the intervention of symmetric functions of residues, in order to pull up results from genus 0 we want $s_2(\omega_{h_i,\pi_0^*(\varphi)}) = s_2(\omega_{\mathbf{h}_i,\varphi})$ in κ. This is a special case of:

Lemma 6.11. *Let $\pi : \overline{C} \to \overline{C}'$ be a finite generically-étale map of degree r between proper smooth geometrically-connected curves over a perfect field k, and let $n > 0$ be a positive integer. Assume $\pi^{-1}(\infty) = r \cdot \xi$ for rational points $\infty \in \overline{C}'(k)$ and $\xi \in \overline{C}(k)$. Let $\operatorname{Spec} A = \overline{C} - \{\xi\}$ and $\operatorname{Spec} A' = \overline{C}' - \{\infty\}$. Pick a nonzero $h \in A[T]$ and define $\mathbf{h} = \mathrm{N}_{A/A'}(h) \in A'[T]$.*

Define $\omega_{h,a} = (\partial_T h)(a^n)a^{n-1}\,\mathrm{d}a/h(a^n)$ and $\omega_{\mathbf{h},a'} = (\partial_T \mathbf{h})(a'^n)a'^{n-1}\,\mathrm{d}a'/\mathbf{h}(a'^n)$ for any $a \in A$ and $a' \in A'$ with $h(a^n)$ and $\mathbf{h}(a'^n)$ nonzero. If $\varphi \in A'$ has the property that $\mathbf{h}(\varphi^n) \in A'$ is nonzero and squarefree then $s_2(\omega_{h,\pi^*\varphi}) = s_2(\omega_{\mathbf{h},\varphi})$ in k.

Proof. We may assume k is algebraically closed. Since $\mathrm{N}_{A/A'}(h(\pi^*(\varphi^n))) = \mathbf{h}(\varphi^n)$ is nonzero and squarefree in A', the physical zeros of $h(\pi^*(\varphi)^n)$ on $\operatorname{Spec} A$ are simple and lie in distinct fibers of π, with π étale at each such zero. Thus, for each zero $x' \in \operatorname{Spec} A'$ of $\mathbf{h}(\varphi^n)$ there is a unique zero x of $h(\pi^*(\varphi)^n)$ in $\pi^{-1}(x')$. We have a factorization $\pi^*(\mathbf{h}) = h \cdot \widetilde{h}$ in $A[T]$, and so

$$\pi^*(\omega_{\mathbf{h},\varphi}) = \omega_{h,\pi^*\varphi} + \omega_{\widetilde{h},\pi^*\varphi}. \tag{6.3}$$

If x is a zero of $h(\pi^*(\varphi)^n)$ and $x' = \pi(x)$ then $\widetilde{h}(\pi^*(\varphi)^n)$ is nonvanishing at x and π is étale at x, so (6.3) implies $\mathrm{Res}_x(\omega_{h,\pi^*\varphi}) = \mathrm{Res}_{x'}(\omega_{\mathbf{h},\varphi})$. Thus, if $Z \subseteq \operatorname{Spec} A$ and $Z' \subseteq \operatorname{Spec} A'$ are the respective zero-loci of $h(\pi^*(\varphi)^n)$ and $\mathbf{h}(\varphi^n)$ then these loci contain all respective poles of $\omega_{h,\pi^*\varphi}$ and $\omega_{\mathbf{h},\varphi}$ away from ξ and ∞, and π induces a residue-preserving bijection from Z to Z'. The respective residues at ξ and ∞ are determined by the residues along Z and Z' by the residue formula, so $s_2(\omega_{h,\pi^*\varphi}) = s_2(\omega_{\mathbf{h},\varphi})$ (and likewise with any higher symmetric function of the residues indexed by the geometric poles). ∎

By Lemma 6.11, we get

$$(-1)^{\mathrm{Tr}_{\kappa/\mathbf{F}_2}(s_2(\omega_{h_i,\varphi}))}\mu(A/(h_i(\pi_0^*(\varphi)^2))) = (-1)^{\mathrm{Tr}_{\kappa/\mathbf{F}_2}(s_2(\omega_{\mathbf{h}_i,\varphi}))}\mu(\kappa[u]/(\mathbf{h}_i(\varphi^2))). \tag{6.4}$$

By the genus-0 version of Theorem 6.8 in [3, Thm. 5.10], for sufficiently large d (with largeness depending only on g, r, and $\deg_{u,T} f$) the right side of (6.4) only depends on φ modulo the reduction of $\mathscr{I}_{\mathbf{H}_i}$ and on d modulo 4, with the congruence on d relaxed to $d \bmod 2$ if $\deg h_i = \deg h$ is even and relaxed to no dependence on large d when $[\kappa : \mathbf{F}_2]$

is even or $4|\deg h$. (There is a minor technical point: the hypothesis of equality of total degrees in [3, Thm. 5.10] may not hold for $\mathbf{H}_i$ and $\mathbf{h}_i$, as we have noted earlier. The only purpose of that hypothesis was to have certain lower bounds in the conclusion be determined by the total degree of the input polynomial in $\kappa[u][T^2]$ in characteristic 2. Since the possible excess of $\deg_{u,T}\mathbf{H}_i$ beyond $\deg_{u,T}\mathbf{h}_i$ is bounded in terms of r, d_i, and $\deg_{u,T} h$, and the choice of d_i can be bounded in terms of r and g, the above conclusion via [3, Thm. 5.10] is therefore nonetheless true.)

Step 4. We are now in position to use the genus-0 results in [3] to show that the quadratic character of $\beta_m/\beta_{m'} \in W^\times$ is trivial for sufficiently large m and m' with $m' \equiv m \bmod 4r$ (and we can use weaker congruence conditions on m and m' when $[\kappa : \mathbf{F}_2]$ or $\deg h$ are even, or when $4|\deg h$). Pick $\delta \gg 0$ satisfying $\delta \equiv d \bmod 4$ (or merely $\delta \equiv d \bmod 2$ when $\deg h$ is even, or no congruence condition when $[\kappa : \mathbf{F}_2]$ is even or $4|\deg h$). Pick $\psi \in \kappa[u]$ of degree δ with $\psi \equiv \varphi \bmod I_i$; such ψ can be found as long as $d, \delta \geq \dim_\kappa(\kappa[u]/I_i)$, and it is automatic that $\mathbf{h}_i(\psi^2)$ is squarefree (that is, $\mathbf{h}_i(\psi^2)$ has nonvanishing Möbius value) since the reduction of $\mathscr{I}_{\mathbf{H}_i}$ into $\kappa[u]$ divides I_i. These largeness conditions on δ only depend on r, the genus, and $\deg_{u,T} f$ (since $\dim_\kappa(\kappa[u]/I_i)$ can be bounded in terms of these parameters).

Choose $\Phi \in W[u]$ of degree d with unit leading coefficient and reduction $\varphi \in \kappa[u]$, so $a_i^2\pi^*(\Phi^2) \in \underline{\mathscr{V}}^0_{2(d_i+rd)}(W)$. Since d and δ are large and $W[u]/\mathscr{I}_i$ is finite and flat over W, we can choose $\Psi \in W[u]$ of degree δ with unit leading coefficient and reduction $\psi \in \kappa[u]$ such that $\Psi \equiv \Phi \bmod \mathscr{I}_i$. We now use (6.1) for h_i with the admissible lift H_i and for $\Phi, \Psi \in W[u] \subseteq \mathscr{A}$ (in the role of $\widetilde{a}$) lifting $\varphi, \psi \in \kappa[u] \subseteq A$ (in the role of a) with the inclusions $W[u] \hookrightarrow \mathscr{A}$ and $\kappa[u] \hookrightarrow A$ defined by π^* and π_0^*. Forming ratios between (6.4) for φ and its analogue for ψ yields $\chi(\beta_{d_i+rd}/\beta_{d_i+r\delta}) = 1$ since $\pi^*\Psi \equiv \pi^*\Phi \bmod \mathscr{I}_{H_i}$ (due to the definition of $\mathscr{I}_i$). This proves that for large m (only depending on r, the genus, and $\deg_{u,T} f$) the class of $\beta_m \in W^\times/(W^\times)^2$ only depends on $m \bmod 4r$ (or merely on $m \bmod 2r$ when $\deg h$ is even, resp. on $m \bmod r$ when $[\kappa : \mathbf{F}_2]$ is even or $4|\deg h$). Running through the same argument with an odd $r' \geq 2g+1$ satisfying $\gcd(r, r') = 1$, possibly replacing κ with an odd-degree extension in the process, we get the same conclusion with r' replacing r. Thus, an application of the Chinese remainder theorem completes the proof of Theorem 6.8. ■

Let us record the key fact shown in the preceding proof:

Corollary 6.12. *With notation and hypotheses as above, $\beta_d/\beta_{d'} \in (W^\times)^2$ for all sufficiently large d and d' (only depending on g and $\deg_{u,T} f$) such that $d \equiv d' \bmod 4$. The congruence condition may be taken modulo 2 if $\deg h$ is even, and it may be dropped if $[\kappa : \mathbf{F}_2]$ is even or $4|\deg h$.*

As an application of our periodicity results for the Möbius function in characteristic 2, we get an analogue of Theorem 1.1 in characteristic 2:

Theorem 6.13. *Assume $p = 2$ and let $f \in A[T^4]$ be squarefree in $K[T]$ with positive degree such that for all $c \in C = \mathrm{Spec}(A)$ the restriction $f_c \in \kappa(c)[T]$ does not vanish as a function on $\kappa(c)$. Define the nonzero ideal $I_{f,\kappa} \subseteq A$ as in Definition 6.4.*

Let J be any nonzero ideal in A that is a multiple of $\mathrm{Rad}(I_{f,\kappa})$. For large n, the function

$$n \mapsto \frac{\sum\limits_{\deg a=n,(f(a),J)=1} \mu(f(a))}{\sum\limits_{\deg a=n,(f(a),J)=1} |\mu(f(a))|} \tag{6.5}$$

is periodic in $n \gg 0$ with period dividing 2. If $[\kappa : \mathbf{F}_2]$ is even or $8|\deg_T f$ then the period for large n is 1. The largeness in n only depends on g, $\deg_{u,T} f$, and $\dim_\kappa(A/J)$.

If J_1 and J_2 are two nonzero multiples of $\mathrm{Rad}(I_{f,\kappa})$ then the functions defined by (6.5) for $J = J_1$ and $J = J_2$ coincide for large n, with largeness that only depends on g, $\deg_{u,T} f$, and the $\dim_\kappa(A/J_i)$'s.

Recall that if f has leading coefficient that has no double zeros on C and $f = h(T^2)$ then Corollary 6.7 gives an "equicharacteristic" formula for $\mathrm{Rad}(I_{f,\kappa})$.

Proof. Since $f(T) = h(T^2)$ with h a polynomial in T^2, the trace term in the exponent in (6.1) vanishes in this case. Thus, it is straightforward to carry over the proof of Theorem 3.1 to the case of characteristic 2 by using (6.1) in the role of (3.3) and using Corollary 6.12 in the role of [5, Thm. 3.6], as follows.

Pick an admissible triple $(\overline{\mathscr{C}}, \widetilde{\xi}, H)$ over $W = W(\kappa)$ and a W-basis $\widetilde{\varepsilon}$ of the coordinate ring of $\overline{\mathscr{C}} - \widetilde{\xi}$ as in §5. Using notation as in the discussion preceding Remark 6.5, it follows from Corollary 6.12 that for sufficiently large n with a fixed parity, the algebraic map $\langle \beta_n \mathcal{L} \rangle_2 : U \to \mathbf{A}^1_\kappa$ is independent of n and $\widetilde{\varepsilon}$ up to adding a constant of the form $c^2 - c$ for some $c \in \kappa$; write L_σ to denote such a map with $(-1)^n = \sigma$. The map

$$\mathrm{Tr}_{\kappa/\mathbf{F}_2} \circ L_\sigma : U(\kappa) \to \mathbf{F}_2$$

is *intrinsic* to the quotient algebra A/I_H because the value of the function $(-1)^{\mathrm{ord}_\xi(\mathrm{lead}(h))} \cdot (-1)^{\mathrm{Tr}_{\kappa/\mathbf{F}_2} \circ L_\sigma}$ on $a \bmod I_H \in U(\kappa) \subseteq A/I_H$ is equal to $\mu(h(a^2))$ for a in a residue class of A/I_H on whose representatives the values of $h(T^2)$ in A are squarefree. Thus, if we only consider large n with such a fixed parity then the value of (6.5) with $J = I_H$ is

$$(-1)^{\mathrm{ord}_\xi(\mathrm{lead}(f))} \cdot \frac{\sum_{a \in U(\kappa)} (-1)^{\mathrm{Tr}_{\kappa/\mathbf{F}_2}(L_\sigma(a))}}{\# U(\kappa)}, \tag{6.6}$$

and likewise (for the same large n) after any finite extension of the constant field. This expression is intrinsic to A/I_H and only involves the large n through its parity, so we deduce the asserted periodic dependence on n mod 2 for uniformly large n when using $J = I_H$. Due to the *intrinsic* nature of (6.6) once I_H is given, the method as in the case $p \neq 2$ carries over to handle both the periodicity and "independence of J" aspects for more general J. ∎

In our asymptotic study of odd characteristic in §3, we saw that for "generic" $f \in A[T^p]$ (*i.e.*, those f such that some exponent $e_x = \ell(\mathscr{O}_{B,x})$ in [5, (3.14), Thm. 4.5] is odd) the length-4 periodic sequence of values $\Lambda_{\kappa' \otimes_\kappa A}(f; n)$ for large n converges to the constant sequence $\{1, 1, 1, 1\}$ as $[\kappa' : \kappa] \to \infty$, and that in general for each fixed congruence class mod 4 the periodic value of $\Lambda_{\kappa' \otimes_\kappa A}(f; n)$ on that congruence class for large n converges to 0, 1, or 2. We wish to prove a similar result for $p = 2$, at least in the case $f \in A[T^4]$.

Assume $p = 2$ and let $f \in A[T^4]$ be as in Theorem 6.13. For each sign $\sigma = \pm 1$ we let $L_\sigma : U \to \mathbf{A}^1_\kappa$ be the map as in the proof of Theorem 6.13. Let $\lambda_{\kappa'}(f, \sigma)$ be the common value of $\Lambda_{\kappa' \otimes_\kappa A}(f; n)$ for all large n with $(-1)^n = \sigma$. As $[\kappa' : \kappa] \to \infty$, the asymptotic behavior of $\lambda_{\kappa'}(f; \sigma)$ (or equivalently the Möbius average (6.5) in degree n, formed with $\kappa' \otimes_\kappa A$) is governed by the degree-2 Artin–Schreier covering $V_\sigma : y^2 - y = L_\sigma$ over the smooth geometrically integral κ-scheme U. There are three mutually exclusive possibilities: V_σ is geometrically connected over κ, V_σ is connected but geometrically disconnected over κ, or V_σ is disconnected. Since the difference function $L_1 - L_{-1} : U \to \mathbf{A}^1_\kappa$ is a constant function, geometric connectivity for V_1 over κ is equivalent to geometric connectivity for

V_{-1} over κ. In view of how L_1 and L_{-1} were constructed, the case of geometric connectivity seems to be the "generic" case (as we vary f); however, unlike the case of odd characteristic, we do not know a convenient criterion (*e.g.*, in terms of the branch multiplicities $\ell(\mathscr{O}_{B,x})$) that is sufficient to ensure geometric connectivity.

Theorem 6.14. *For $p = 2$ and $f \in A[T^4]$ as in Theorem 6.13, let $d_0 = -\mathrm{ord}_\xi(\mathrm{lead}(f))$ and fix $\sigma = \pm 1$. If the Artin-Schreier double cover V_σ of U as above is geometrically connected then $\lambda_{\kappa'}(f;\sigma) \to 1$ as $[\kappa' : \kappa] \to \infty$. If V_σ is disconnected then $\lambda_{\kappa'}(f;\sigma) = 1 - (-1)^{d_0}$ for all κ'/κ. Finally, if V_σ is connected but geometrically disconnected then $\lambda_{\kappa'}(f;\sigma) = 1 - (-1)^{d_0}$ if $[\kappa' : \kappa]$ is even and $\lambda_{\kappa'}(f;\sigma) = 1 + (-1)^{d_0}$ if $[\kappa' : \kappa]$ is odd.*

Proof. If the double cover $V_\sigma \to U$ is connected but geometrically disconnected over κ then it splits over even-degree extensions but not over odd-degree extensions. Since $\lambda_{\kappa'}(f;\sigma)$ is given by 1 minus the expression in (6.6) with κ' replacing κ, to handle the cases when V_σ is not geometrically connected we must show that if V_σ is disconnected (resp. connected but geometrically disconnected) then (6.6) is equal to $(-1)^{d_0}$ (resp. $-(-1)^{d_0}$). In the disconnected case we must have $L_\sigma = \phi^2 - \phi$ for some global function ϕ on the normal variety U, so this case is obvious. In the connected but geometrically disconnected case it follows from the geometric connectivity of U over κ and the standard short exact sequence

$$1 \to \pi_1(U_{\overline{\kappa}}) \to \pi_1(U) \to \mathrm{Gal}(\overline{\kappa}/\kappa) \to 1$$

of étale fundamental groups that the cohomology class of the $\mathbf{Z}/2\mathbf{Z}$-torsor $V_\sigma \to U$ is "the same" as that of its fiber over any point $a \in U(\kappa)$. Hence, all such fibers

$$\mathrm{Spec}\,\kappa[y]/(y^2 - y - L_\sigma(a))$$

are connected, so by Artin–Schreier theory $L_\sigma(a) \in \kappa$ has non-vanishing $\mathbf{F}_2$-trace for all $a \in U(\kappa)$. This gives that (6.6) is equal to $-(-1)^{d_0}$ as desired.

It remains to consider the case when V_σ is geometrically connected. For each κ'/κ and $a \in U(\kappa')$, the sign $(-1)^{\mathrm{Tr}_{\kappa/\mathbf{F}_2}(L_\sigma(a))}$ is equal to 1 (resp. -1) when the finite étale degree-2 connected covering $V_{\sigma/\kappa'} \to U_{\kappa'}$ has a-fiber that is disconnected (resp. connected), so (6.6) with κ' in the role of κ is $(-1)^{d_0}(\#V_\sigma(\kappa') - \#U(\kappa'))/\#U(\kappa')$. Hence, exactly as in the case of odd characteristic, by geometric connectivity of V_σ and U we may infer from the Lang–Weil estimate that this ratio approaches 0 (and so $\lambda_{\kappa'}(f;\sigma) \to 1$) as $[\kappa' : \kappa] \to \infty$. ■

By adapting the proof of Theorem 3.8, an argument with quadratic (Artin-Schreier) character sums gives non-triviality of characteristic-2 correction factors over large finite fields when f is fixed:

Theorem 6.15. *For $p = 2$ and $f \in A[T^4]$ as in Theorem 6.13, $\lambda_{\kappa'}(f;\sigma) \neq 1$ for all extensions κ'/κ with sufficiently divisible degree (depending only on the total degree of f).*

Before we prove Theorem 6.15, we make some remarks. Of course, by Theorem 6.14 we only need to do some work in the case that the double covers $V_{\pm 1}$ are geometrically connected. Also, though some aspects of our treatment of non-triviality in characteristic 2 will be more complicated than in our earlier work for odd p (due to the use of truncated Witt vectors), there is an important simplification: the intervention of $e_n \bmod 2$ in the case of odd p does not arise in the case $p = 2$. Finally, we note that it is unclear how to prove analogues of Theorems 3.6 and 3.8 for families of f's with $p = 2$ (though we are sure that reasonable analogues must hold) because we lack a convenient sufficient criterion for geometric connectedness of $V_{\pm 1}$ in terms of discrete invariants of f (as we have when $p \neq 2$ via oddness of the multiplicity at some point on the branch scheme B of $Z_f \to \mathbf{A}^1$).

Proof. To adapt the method of proof of Theorem 3.8 in the case $p = 2$ with a fixed f, the main issue is to handle truncated Witt vectors as in the discussion preceding Remark 6.5. The first step is to eliminate the intervention of the mysterious unit β_n that arises in the definition of the algebraic function $L_\sigma = \langle \beta_n \mathfrak{L} \rangle_2$ on U for n such that $\sigma = (-1)^n$. ■

Appendix A. Numerical testing

In this appendix we address the testing of the conjectural asymptotic (2.4). As in [3], we have not made error estimates to justify our data rigorously.

Our examples will be affine curves given by the complement of a κ-rational point in a smooth hyperelliptic curve. More specifically, for $g \geq 1$ we use affine curves of the form

$$C : y^2 + c_0(x)y = c_1(x), \tag{A.1}$$

where $c_0(x)$ and $c_1(x)$ are in $\kappa[x]$ with

$$\deg(c_0(x)) \leq g, \quad \deg(c_1(x)) = 2g + 1,$$

and we require $c_0(x) \neq 0$ when $p = 2$. These conditions guarantee that in odd characteristic $c_0(x)^2 + 4c_1(x)$ has odd degree (so it is nonzero and not a square), whence (A.1) is geometrically integral in odd characteristic. The degree conditions force (A.1) to have no solution $y \in \overline{\kappa}(x)$ in characteristic 2, so geometric integrality holds in all characteristics. The affine curve (A.1) is smooth if and only if there is no point (x_0, y_0) on the curve satisfying the two conditions

$$2y_0 + c_0(x_0) = 0, \quad c_0'(x_0)y_0 = c_1'(x_0).$$

Let us suppose c_0 and c_1 are chosen so that C is smooth, and let

$$A = \kappa[C] = \kappa[x, y]/(y^2 + c_0 y - c_1).$$

Elements of A can be uniquely written as

$$a = a_0(x) + a_1(x)y,$$

where $a_0(x)$ and $a_1(x)$ are in $\kappa[x]$. The equation (A.1) has a singular point on the line at infinity in $\mathbf{P}^2_\kappa$ when $g \geq 2$, and for all $g \geq 1$ the smooth compactification of C has genus g and exactly one (κ-rational) geometric point ξ at infinity.

Our method of estimating $C_A(f)$ and calculating $\Lambda_A(f; n)$, two terms which appear in (2.4), differs in the higher genus setting from the procedure we followed in genus 0 in [3]. Let us explain the difference. In [3], we accurately estimated the constant $C_{\kappa[u]}(f)$ by modifying its definition as an infinite product in a way that sped up convergence of the product. For practical purposes we use a method of estimating $C_A(f)$ in higher genus that avoids infinite products at the expense of less rigor in the numerical verification: we exploit a connection between separable and inseparable irreducible polynomials, as follows. Let $f(T) \in A[T^p]$ be irreducible in $K[T]$ without local obstructions. Write $f(T) = F(T^{p^m})$ with $m \geq 1$ as big as possible, so $F(T) \in A[T]$ is irreducible and separable in $K[T]$ with no local obstructions over A. Note $C_A(f) = C_A(F)$. Since $F(T)$ is irreducible and separable in $K[T]$, we believe $F(T)$ satisfies (2.3). Granting this, we can get an estimate for $C_A(F)$ by computing all parts of (2.3) other than $C_A(F)$ as n grows. Then we use this as our estimate for $C_A(f)$ in testing (2.4).

Now we turn to the calculation of $\Lambda_A(f;n)$ in the hyperelliptic case, where A is presented as a degree-2 finite flat extension of $\kappa[x]$. We also assume that the leading coefficient of f in A has at worst simple zeros on C. If $p \neq 2$ then we let $I = I_f \subseteq A$ denote the nonzero radical ideal as in §2: its zero locus on C is the image in C of the finite branch scheme for the generically étale projection $Z_f \to \mathbf{A}^1_\kappa$ (where $Z_f \subseteq C \times \mathbf{A}^1_\kappa$ is the zero scheme of f). If $p = 2$ and $f \in A[T^4]$, then we let $I = I_{f,\kappa} \subseteq A$ be the nonzero (possibly nonradical) ideal as in Definition 6.4. By Corollary 6.7, since the leading T-coefficient of f has at worst simple zeros on C it follows that $\mathrm{Rad}(I_{f,\kappa}) = I_h$ where $f = h(T^2)$. For the purposes of computing the periodic tail of the values $\Lambda_A(f;n)$ as in (2.2) (see Theorem 6.13 for the periodicity in case $p = 2$), we just need to know $\mathrm{Rad}(I_{f,\kappa})$ rather than $I_{f,\kappa}$.

Hence, over finite fields of any characteristic we are motivated to address the problem of computing the ideal I_f when $f \in A[T^p]$ is squarefree in $K[T]$ and primitive with respect to A such that $\mathrm{lead}(f) \in A$ has at worst simple zeros on C. For such f we shall give a formula for I_f in terms of resultants. To establish such a formula it is convenient (as in [5, §2]) to permit $C = \mathrm{Spec}\, A$ to have no restrictions on its locus at infinity and to permit the base field to be an arbitrary perfect field with characteristic $p > 0$; note that the definition of I_f makes sense without any restrictions at infinity and for any such perfect base field. We write $R_A(h_1, h_2) \in A$ to denote the resultant of $h_1, h_2 \in A[T]$ (taken to be 0 if some h_j vanishes). A preliminary "formula" for I_f is given by:

Theorem A.1. *Assume that* $\mathrm{lead}(f) \in A - \{0\}$ *has at worst simple zeros on* C*. Let* $\mathfrak{I}$ *be the ideal generated by the resultants* $R_A(f, \partial f) \in A$ *as* ∂ *runs over all* κ*-derivations* $A \to A$ (*extended to* $\kappa[T]$*-derivations on* $A[T]$ *by acting on coefficients*)*. Then* $I_f = \mathrm{Rad}(\mathfrak{I})$.

Proof. Since $\Omega^1_{A/\kappa}$ is locally free of rank 1 and the formation of $R_A(f, \partial f)$ is local on $\mathrm{Spec}\, A$, the identity $R_A(f, a \cdot \partial f) = a^{\deg_T f} R_A(f, \partial f)$ shows that the problem is Zariski-local on C. Our problem is one of comparing two ideals in a Dedekind domain. Let $Z \subseteq C \times \mathbf{A}^1_k$ be the zero scheme of f, and let pr_1 and pr_2 denotes its projections to C and $\mathbf{A}^1_k$ respectively. Also let $B \subseteq Z$ be the finite branch scheme for the generically étale projection $\mathrm{pr}_2 : Z \to \mathbf{A}^1_k$.

Localizing to the case when $\Omega^1_{A/k}$ is free of rank 1 with basis corresponding to a k-linear derivation $D : A \to A$, we want to show that $R_A(f, Df)$ generates $\mathfrak{I}$. We may replace the base field with an algebraic closure since the formation of $\mathfrak{I}$ is compatible with change in the base field. In other words, now we take the base field to be an algebraically closed field k of characteristic p. As a first step toward proving $\mathfrak{I} = (R_A(f, Df))$ in A, we show that the k-finite closed subschemes $\mathrm{Spec}(A/(R_A(f, Df)))$ and $\mathrm{Spec}(A/\mathfrak{I})$ in C have the same support. Since $\mathrm{lead}(f)$ has simple zeros, for each $c \in C(k)$ at least one of the specializations f_c or $(Df)_c = D_c(f_c)$ in $k(c)[T] = k[T]$ has the same T-degree as f or Df respectively. Hence, $R_A(f, Df)$ has a zero at c if and only if the specializations f_c and $D_c(f_c)$ have a common zero at some $t \in k$, which certainly forces $(c, t) \in C \times \mathbf{A}^1_k$ to lie on the zero scheme $Z = Z_f$ of f on $C \times \mathbf{A}^1_k$. Meanwhile, c is a zero of $\mathfrak{I}$ if and only if there is some point $(c, t') \in Z$ at which $\mathrm{pr}_2 \colon Z \to \mathbf{A}^1_k$ is not étale. Thus, it suffices to prove that for any point $z = (c, t) \in Z$, the specializations f_c and $(Df)_c$ in $k(c)[T]$ vanish at $t \in k$ if and only if $\mathrm{pr}_2 \colon Z \to \mathbf{A}^1_k$ is non-étale at z. This equivalence was established in the proof of [5, Thm. 2.5]. Thus, $R_A(f, Df) \neq 0$ and, since A is Dedekind, it remains to compare the k-lengths of the artin local rings of $\mathrm{Spec}(A/(R_A(f, Df)))$ and $\mathrm{Spec}(A/\mathfrak{I})$ at each $c \in C$ that lies in their common support.

Fix such a point $c \in C$ and choose an isomorphism $\widehat{\mathscr{O}}_{C,c} \simeq k[\![u]\!]$. This carries D over to a unit multiple of ∂_u. The finite-length stalk of $\Omega^1_{Z/\mathbf{A}^1_k}$ at a point $(c,t) \in Z$ is isomorphic to its completion, which is $(k[\![u, T-t]\!]/(f, \partial_u f))\mathrm{d}u$. Thus, we need to show

$$\text{(A.2)} \qquad \mathrm{ord}\, R_{k[\![u]\!]}(f, \partial_u f) \stackrel{?}{=} \sum_t \dim_k k[\![u, T-t]\!]/(f, \partial_u f),$$

where the sum runs over the finitely many $t \in k$ such that $(c,t) \in B$ (the dimension term is zero at other $t \in k$). Viewing f in $k[\![u]\!][T]$, the proof of [5, Thm. 2.5] shows that $(c,t) \in Z$ lies in B if and only if $f_c(t)$ and $(\partial_u f)_c(t)$ vanish. Thus, closely approximating $k[\![u]\!]$-coefficients of f by elements of $k[u]$ without a common factor does not change either side of (A.2) and provides an element $\widetilde{f}$ in $k[u, T^p]$ satisfying our basic assumptions from the start in the case of the affine base curve $\mathrm{Spec}\, k[u]$. In this way, we are reduced to the case of plane curves.

It now suffices to show that if $\{f(u,T) = 0\}$ and $\{h(u,T) = 0\}$ are (possibly empty) plane curves over an algebraically closed field K such that $\mathrm{lead}(f), \mathrm{lead}(h) \in K[u]$ do not have a common zero and if these plane curves have no common irreducible components then

$$\text{(A.3)} \qquad \mathrm{ord}_0\, R_{K[u]}(f, h) = \sum_{t \in K} i_{(0,t)}(f, h),$$

where $i_{(0,t)}(f,h) = \dim_K \widehat{\mathscr{O}}_{\mathbf{A}^2_K,(0,t)}/(f,h)$ is the intersection number. This is Zeuthen's rule; see [3, Lemma 4.6] for a proof of Zeuthen's rule in arbitrary characteristic. ■

Suppose $\Omega^1_{A/k}$ is free with basis corresponding to a k-linear derivation $D : A \to A$. It follows from Theorem A.1 and the formula $R_A(f, a\partial f) = a^{\deg_T f} R_A(f, \partial f)$ that $\mathcal{I}$ is principal with generator $\mathrm{Rad}(R_A(f, Df))$ if $\mathrm{lead}(f) \in A - \{0\}$ has at worst simple zeros on C. This applies when $A = \kappa[u]$ by taking $D = \partial_u$, thereby recovering [5, Ex. 2.4] if $\mathrm{lead}_T(f) \in \kappa[u]$ is separable. We now will show that an analogous calculation applies in the hyperelliptic case.

Corollary A.2. *If C is given by* (A.1) *and* $\mathrm{lead}(f) \in A$ *has at worst simple zeros then* $I_f = \mathrm{Rad}(R_A(f, D(f)))$, *where $D\colon A \to A$ is the derivation determined by $D(x) = 2y + c_0(x)$ and $D(y) = c_1'(x) - c_0'(x)y$.*

Proof. The derivation D is the unique extension of $(2y+c_0(x))\partial_x$ on $\kappa[x]$ to A. We show that $\Omega^1_{A/\kappa}$ is free with generator corresponding to D. Choose any κ-linear derivation $\partial : A \to A$. By (A.1),

$$(2y + c_0(x))\partial(y) = (c_1'(x) - c_0'(x)y)\partial(x).$$

The smoothness of C implies that the ideals $(2y + c_0(x))$ and $(c_1'(x) - c_0'(x)y)$ are relatively prime, so $\partial(x) = (2y + c_0(x))h$ and $\partial(y) = (c_1'(x) - c_0'(x)y)h$ for some $h \in A$. Thus $\partial = hD$. ■

In odd characteristic, a change of variables lets us take $c_0(x) = 0$, in which case

$$D(a_0(x) + a_1(x)y) = a_1(x)c_1'(x) + 2a_1'(x)c_1(x) + 2a_0'(x)y.$$

Example A.3. Consider the elliptic curve $y^2 = x^3 - x$ in characteristic $p \neq 2$. We have $D(x) = 2y$ and $D(y) = 3x^2 - 1$. For $f(T) = T^p + a$, where $c \in A$, we have $D(f) = D(a)$. This is in A, so $R(f, D(f)) = (D(f))^{\deg_T f} = (D(a))^p$. Therefore $I = \mathrm{Rad}(D(a))$.

Example A.4. Consider the genus 2 hyperelliptic curve $y^2 = x^5 - x$ in characteristic $p \neq 2$. We have $D(x) = 2y$ and $D(y) = 5x^4 - 1$.

Since we always work in the setup in Corollary A.2, we let $R = R_f$ denote the specific resultant in this corollary. The conditions $(f(a), I_f) = (1)$ for $p \neq 2$ and $(f(a), I_{f,\kappa}) = (1)$ for $p = 2$ are not easy to check directly by computer calculation, but replacing I_f and $I_{f,\kappa}$ with the nonzero multiple $\mathrm{N}_{A/\kappa[x]}(R) \cdot A$ gives the condition $(f(a), \mathrm{N}_{A/\kappa[x]}(R)) = (1)$, or equivalently $(\mathrm{N}_{A/\kappa[x]}(f(a)), \mathrm{N}_{A/\kappa[x]}(R)) = (1)$. This latter condition can be checked in $\kappa[x]$ very quickly on a computer. Thus, in our numerical work with hyperelliptic curves we computed

$$\Lambda_{A,\mathrm{N}(R)}(f;n) = 1 - \frac{\sum\limits_{\deg a = n, (f(a),\mathrm{N}(R))=1} \mu(f(a))}{\sum\limits_{\deg a = n, (f(a),\mathrm{N}(R))=1} |\mu(f(a))|} \tag{A.4}$$

using the norm ideal $\mathrm{N}(R)$ for R as above; by Theorem 3.1 for $p \neq 2$ and Theorem 6.13 for $p = 2$, (A.4) recovers $\Lambda_A(f;n)$ for sufficiently large n.

In our treatment of the numerical examples in [3], we rigorously computed $\mu(f(a))$ for *all* $a \in A = \kappa[u]$. However, in the higher genus case we have not computed explicit general formulas for $\mu(f(a))$ in nontrivial examples, essentially because the proofs of the higher-genus Möbius periodicity theorems are too abstract (whereas the proofs in genus 0 in [3] are more concrete and algebraic, thereby making them effective in specific examples). Consequently, for higher genus we computed $\Lambda_{A,\mathrm{N}(R)}(f;n)$ until we saw a plausible periodic pattern in n mod 4 and then we assumed this to be the true period in our check on (2.4).

We now present our numerical data for polynomials having coefficient rings corresponding to the curves listed in Table 1. We list the affine equation, the constant field, the genus, and the numerator of the zeta function of the affine curve over the constant field.

Equation	Constant Field	Genus	Zeta Numerator
$C_1 : y^2 + xy = x^3 + 1$	$\mathbf{F}_2$	1	$1 + t + 2t^2$
$C_2 : y^2 = x^3 - x$	$\mathbf{F}_3$	1	$1 + 3t^2$
$C_3 : y^2 = x^5 - x$	$\mathbf{F}_3$	2	$1 - 2t^2 + 9t^4$
$C_4 : y^2 = x^5 - x$	$\mathbf{F}_5$	2	$(1 - 5t^2)^2$

TABLE 1. Affine curves

In each example below, when the (apparent) period length for $\Lambda_{A,\mathrm{N}(R)}(f;n)$ exceeds 1 we list as the first term of the (apparent) periodic sequence of values the value that occurs for $n \equiv 1 \bmod 4$.

Example A.5. Let $f(T) = xT^{16} + y^2T^8 + xy$ in $\mathbf{F}_2[C_1][T]$, where C_1 is the first curve in Table 1. For $n \geq 1$ define $\Lambda_{A,\mathrm{N}(R)}(f;n)$ as in (A.4). For $8 \leq n \leq 18$, we computed $\Lambda_{A,\mathrm{N}(R)}(f;n) = 16/15$.

A check on (2.4) using $f(T)$ is given in Table 2. The second column in Table 2 gives the number of occurrences of primality for the ideal $(f(a))$ when $\deg a = n$. The third column gives the estimate coming from the right side of (2.4).

Example A.6. Let $f(T) = T^9 + T^6 + 2x^3 + x^2 + 2$ in $\mathbf{F}_3[C_2][T]$. For $4 \leq n \leq 14$, $\Lambda_{A,\mathrm{N}(R)}(f;n)$ has the periodic pattern $8/9, 1, 10/9, 1$. (The $8/9$ occurs for $n \equiv 1 \bmod 4$.) We use this as the periodic pattern for all n. See Table 3.

n	Count	Estimate	Ratio
10	6	4.25	1.41
11	10	7.70	1.30
12	19	14.08	1.35
13	19	25.90	0.73
14	46	48.07	0.96
15	80	89.91	0.89
16	164	168.66	0.97
17	317	317.66	1.00
18	584	600.26	0.97
19	1138	1137.70	1.00
20	2162	2162.25	1.00
21	4118	4119.85	1.00
22	7842	7867.34	1.00
23	14962	15054.21	0.99
24	28768	28860.23	1.00
25	55315	55423.28	1.00
26	106420	106603.61	1.00

TABLE 2. $xT^{16} + y^2T^8 + xy$ over $\mathbf{F}_2[C_1]$

n	Count	Estimate	Ratio
5	2	2.848	0.70
6	12	8.010	1.50
7	20	22.843	0.88
8	70	54.023	1.30
9	118	128.160	0.92
10	338	389.286	0.87
11	1152	1179.646	0.98
12	2959	2919.645	1.01
13	7040	7186.849	0.98
14	22674	22522.963	1.01
15	70162	70071.480	1.00
16	177207	177368.456	1.00

TABLE 3. $T^9 + T^6 + 2x^3 + x^2 + 2$ over $\mathbf{F}_3[C_2]$

In the next two examples for curves with genus > 1 we work with $n \geq 4$ because smaller values of n are not values of $\deg = -\mathrm{ord}_\xi$ on A (due to Weierstrass gaps at ξ).

Example A.7. Let $f(T) = T^3 + x^2y$ in $\mathbf{F}_3[C_3][x]$. For $4 \leq n \leq 12$, $\Lambda_{A,\mathrm{N}(R)}(f;n)$ has the pattern 1,2,1,0. We use this for all n. The exact value of $C_A(f)$ is $(\log 3)(9/8)$. See Table 4.

Example A.8. Let $f(T) = yT^5+x^4+2$ in $\mathbf{F}_5[C_4][x]$. For $4 \leq n \leq 10$, $\Lambda_{A,\mathrm{N}(R)}(f;n)$ has the alternating pattern 1,2. We use this for all n. The exact value of $C_A(f)$ is $(\log 5)(64/125)$. See Table 5.

n	Count	Estimate	Ratio
4	0	0.0	
5	4	4.1	0.988
6	30	20.2	1.485
7	30	26.0	1.152
8	0	0.0	
9	192	182.3	1.053
10	962	984.2	0.977
11	1304	1342.0	0.972
12	0	0.0	
13	10232	10220.0	1.001
14	57042	56940.2	1.001
15	79880	79716.1	1.002
16	0	0.0	

TABLE 4. $T^3 + x^2y$ over $\mathbf{F}_3[C_3]$

n	Count	Estimate	Ratio
4	32	38.14	0.84
5	84	79.63	1.05
6	688	680.92	1.01
7	1589	1490.12	1.07
8	13568	13245.58	1.02
9	29596	29802.32	0.99
10	271812	270930.30	1.00

TABLE 5. $yT^5 + x^4 + 2$ over $\mathbf{F}_5[C_4]$

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, U.S.A.
E-mail address: bdconrad@umich.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, U.S.A.
E-mail address: kconrad@math.uconn.edu

Department of Mathematics, Boston College, Chestnut Hill, MA 02467-3806, U.S.A.
E-mail address: gross@bc.edu